Dynamics of Social Systems

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C O N N E X I O N S

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Chapter 1

On the Eschatology of the Human Condition[1]

1.1 On the Eschatology of the Human Condition

We are now facing a crisis concerning the availability of an assured supply of energy and other resources to feed the ravenous productive capacity of our society. Complicating this critical situation is the problem of controlling the unwanted byproducts of production and consumption. Our present situation stands in striking contrast to the ubiquitous pattern of growth and expansion evidenced by nearly every aspect of what has come to be known as civilization during the past century. We can no longer avoid, nor dare we postpone, the obligation to investigate these issues in a serious and expeditious manner in order to ascertain whether the present reverses are but temporary diversions and fluctuations from a long range pattern of stable and sustainable growth, or whether they are harbingers of limits to expansion imposed by nature, limits which cannot be passed with impunity. Our investigation must, insofar as it is able, be scientific; stripping away the superficialities of mere experience in order to uncover the underlying motive forces which frame and mold the opportunities and constraints whose realizations we recognize in the continuing progression of daily events. As in so many fields whose mysteries have been revealed through diligent application of the methods of science, here too we may expect to find that superficial experience misleads and diverts the attention from the matter of essential significance.

1.1.1 Exponential Growth

In the issue at hand the lesson taught by the immediate experience of life in America and the other industrial nations is that continuing exponential growth, growth which cumulates according to the law of compound interest, growth without limit or constraint, is the natural human condition. The more reflective amongst us may examine the historical record to penetrate beyond the present and the immediate past, but they too find evidence to support the conclusion of immediate experience unless they search so far into the past that the very nature of society seems so different from our own as to invalidate any method or even hope of comparison. Yet the Malthusian critique, in forms more or less sophisticated, remains to haunt us in even the best of times with the suspicion that those early civilizations, so unlike our own, hold the key not only to the flowering of our recent past but also to a withered future.

1.1.2 Growth of human knowledge

There can be little doubt that immediate experience provides a firm foundation for the expectation of continued exponential growth. For example, we derive from human knowledge all our skills and abilities to

[1] This content is available online at <http://cnx.org/content/m17670/1.4/>.

turn the base matter of the world to our own interests. Knowledge grows with time; if we attempt to measure it—not by its essential quality, but by its quantity as manifested in reduction to printed form stored as books and journals in archival libraries—we find that knowledge, too, grows exponentially, apparently inexorably increasing by a fixed fraction year after year. Figure 1 exhibits the growth of the number of scientific journals with time –one typical measure of the growth of knowledge. The vertical axis scale is so arranged that exponential growth is represented by straight lines. The figure suggests that scientific knowledge has grown exponentially for more than 200 years, doubling its quantity every 15 years. If this pattern of growth persists for another two and one half decades, an addition of some 12% to the historic record represented by the figure, there will be in the year 2000 more than 1 million scientific journals publishing more than 25 million scientific articles each year. It has been calculated that each scientific article published today represents an investment of about $25,000; this cost will certainly not decrease in the future. Extrapolation of the historic trend of Figure 1.1 therefore entails the conclusion that annual investment in scientific effort in the year 2000 will reach nearly one trillion dollars, which is approximately the 1973 gross national product of the United States. If the trend continues until 2050, the investment in science will rise to 10 trillion dollars annually. We do not suggest that these estimates are predictions; rather, they have been introduced to provide the reader with a yardstick with which to measure the encouraging projections of the technological optimists who argue that the increased application of novel technology will relax current constraints which manifest themselves in the form of continual shortages rotating from food to productive capacity to energy and back again to food. Newton has already combed the beach, found the smoother pebbles and prettier shells; we must explore his great ocean of truth and the price of the vessel in which we can do this must be paid. If continuance of exponential growth is to depend on technology, and ultimately on science, then the growth of technology and science must themselves continue on their exponential path, and then the projections provided above will, no matter that they boggle common sense, foreshadow reality.

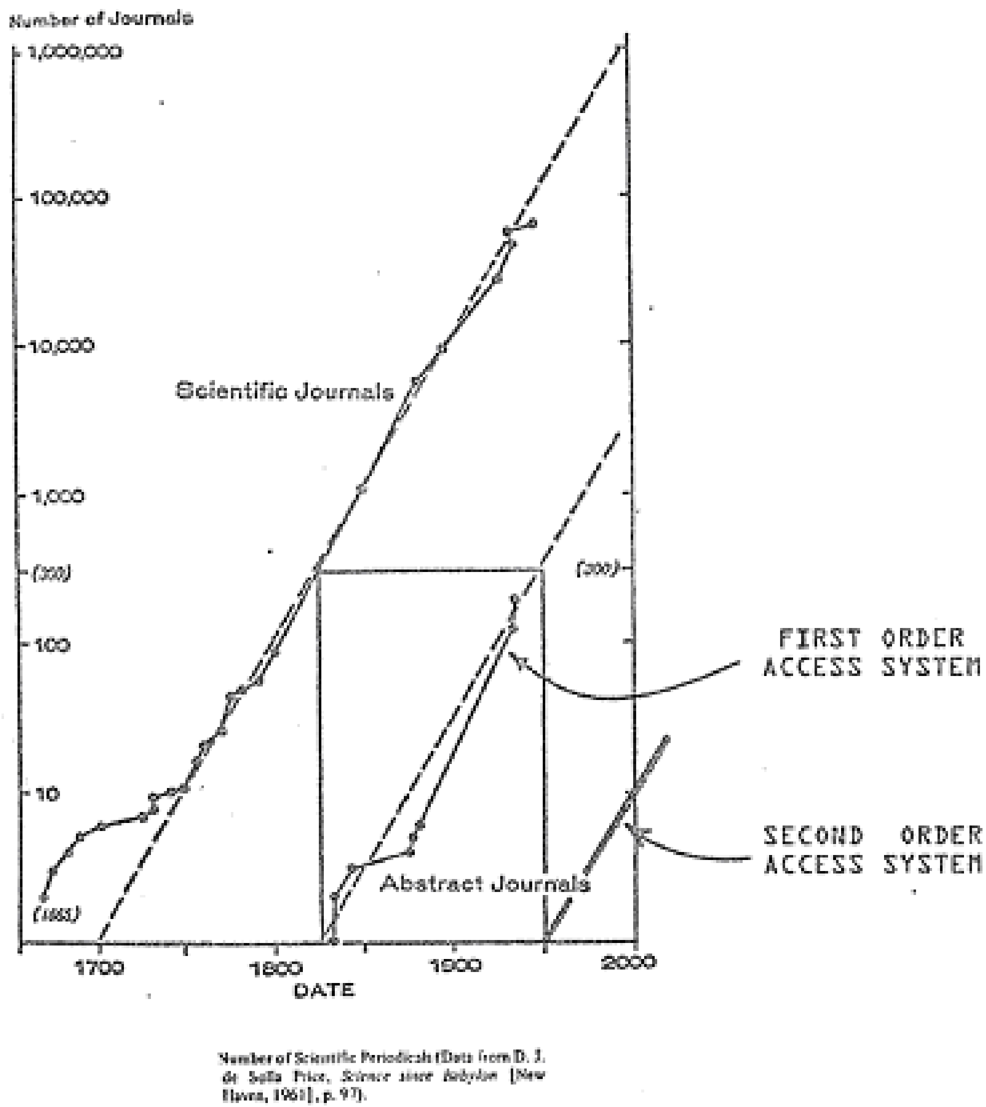
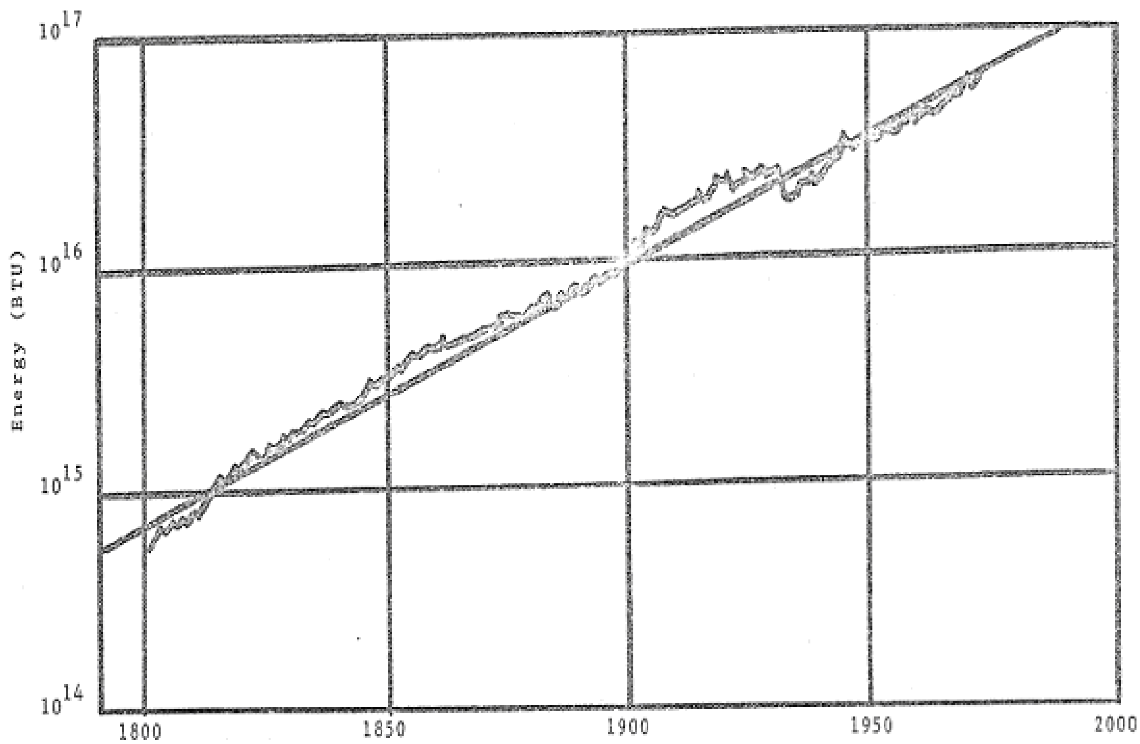


Figure 1.1

1.1.3 Growth of the energy system

The recent historic exponential growth of civilization is more apparent to us all in other ways. To consider a timely example: although the sources of energy used in the United States have changed dramatically since 1800, the growth of annual inputs to the energy system of this country has deviated but little from its exponential trend in the intervening 170 years. There was a slight relative excess from 1900 until the

Great Depression in 1929 and a subsequent defect until the end of World War II, but the deviations from the exponential trend displayed in Figure 1.2 are small when compared with the enormous social dislocations with which they were associated, and they seem not to have any long term effect on the underlying growth pattern.



Inputs to the Energy System of the United States, 1800-1970

Figure 2

Figure 1.2

According to Figure 1.2, inputs to the energy system of the United States have been doubling every 26 years; they "should", if growth were to continue unchecked, double again between the present time (1974) and the year 2000. Thus, could we now provide sufficient energy merely to maintain present consumption levels, by the year 2000 we would find that we would be providing for but one-half our then "normal" requirements, based upon the hypothesis that the historic exponential growth trend in energy utilization reflects a natural and appropriate feature of civilization. Upon this hypothesis it follows that most Americans now alive will live to see the day when society will be able to assuage but half their "natural" craving for energy. On this scale, major oil discoveries such as the Alaskan North Slope field and the North Sea deposits diminish in stature: total North Slope recovery is anticipated to be equivalent to 3 years consumption for the United States at present usage rates. Our ability to provide energy in amounts that will continue to double every

26 years clearly demands major technological innovations and extensive capital investment. Assimilation of the byproducts of these efforts, social as well as substantial, may require still greater efforts and ingenuity.

1.1.4 Population Growth

It is sometimes thought that population growth is the essential driving force behind the general exponential growth of other components of society. That this is not so is readily seen by comparison of the rates of growth of United States population and of the inputs to the energy system of the United States. Recent population growth rates correspond to a doubling period of about 45 years compared with the 26 year doubling period for energy input growth; this simply means that per capita energy inputs have been growing. Nevertheless, population growth is an important component in the general scheme of expansion exhibited by our civilization, and one which affects the life style of individuals in a relatively direct way, for within an adult lifetime of 50 years an American can expect to see the population double (if trends continue). The effect would be a consequent major density increase in urban living areas, increased strains on commodity delivery and other communication systems, increased inequalities in the distribution of wealth, larger average community size, and an increasingly impersonal and depersonalized social life outside the spheres of friendship and work role. Contrast this situation with the life of the typical western European in the Middle Ages, say 700-1100: population growth was negligible during this period; personal mobility was low; and personal associations and interactions remained relatively stable throughout most people's lifespan.

Figure 1.3 shows that the population of the United States has changed in different ways at different times: in the earliest periods after European settlement, growth was exponential and extremely rapid; from 1650 to about 1880, population growth was again exponential with virtually no deviations during this 230 year interval. Since 1880 there has been a marked decline in the rate of growth with irregularities which obscure the general trend features. We may nevertheless conclude that any American born between 1650 and 1850 could confidently conclude from personal experience and the historical record that exponential population growth is a natural feature of life in America. The marked change evident in the manner of growth of population during the period centered about 1880 calls for an explanation) and one is readily forthcoming. Prior to that period, there remained a western frontier which was, bit by bit, continually pushed back thereby effectively increasing the land area of the settled nation, until the constraint of fixed geographical and settled limits induced a change in the nature of population increase. Indeed, during the earlier periods, population increase in the United States did not necessarily lead to increased population density due to the effect of territorial expansion, so that, although more recent periods have seen smaller rates of growth, the local population density experienced by most Americans is probably increasing more rapidly now than before.

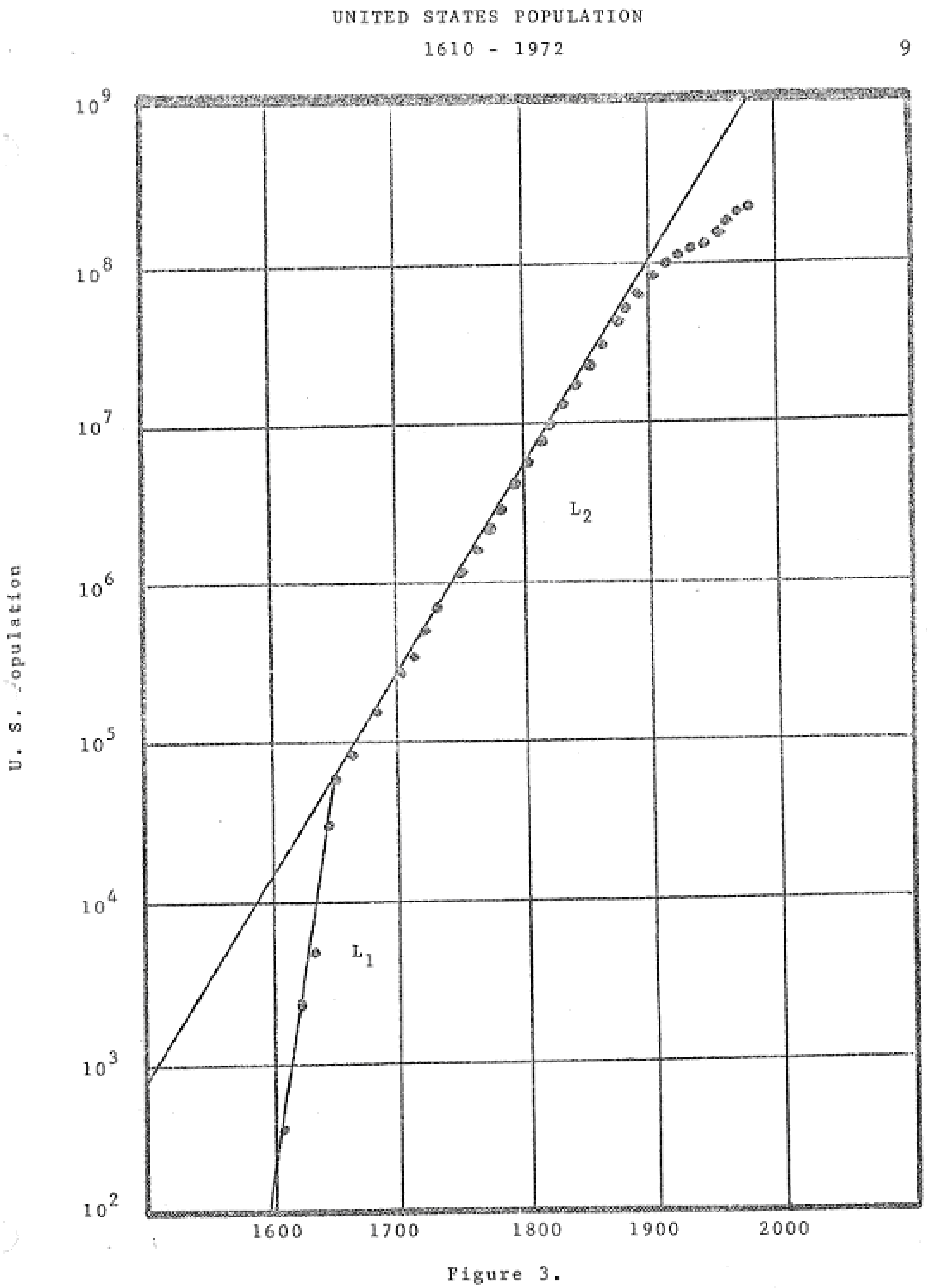

UNITED STATES POPULATION
1610 - 1972
9
U. S. Population
10^9
10^8
10^7
10^6
10^5
10^4
10^3
10^2
L2
L1
1600
1700
1800
1900
2000
Figure 3.

Figure 1.3

1.1.5 Alternative Possibilities

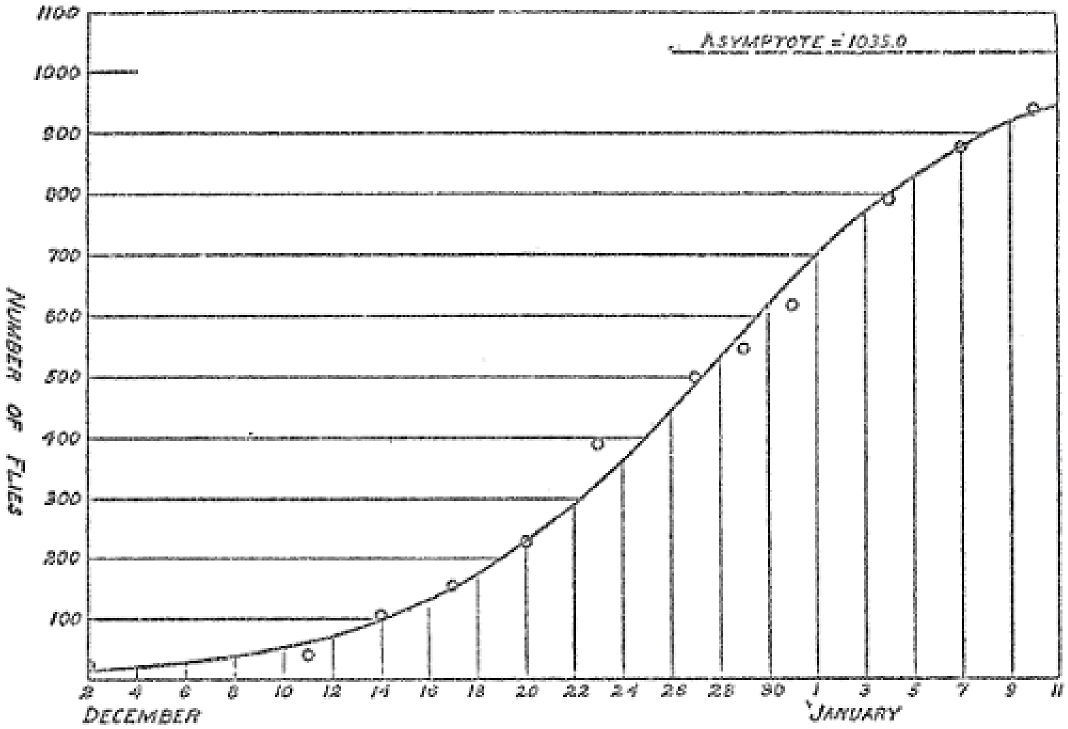
The analysis to this point appears to confirm the generality of exponential growth for various important segments of civilization over periods of time significantly longer than a single generation. The feature of change in the rate of growth of United States population also suggests that there are some mechanisms which can distort or perhaps even destroy the operation of exponential increase. Let us turn our attention to the determination of what these might be and whether they and their effects are intrinsic and unavoidable, or extrinsic and removable.

We would like, of course, to be able to experiment with numerous identical copies of our world with all its inhabitants and curiosities, subjecting each replica to a distinct set of circumstances and following each along its future path to its terminus, thus we could establish the more and the less desirable modes of development which are open to us, asses their benefits and costs, and learn how to direct ourselves and our posterity, if not to the best of all possible worlds, at least away from the worst. That this option is not open to us should not act as a deterrent to serious consideration of the multiple possibilities the future holds, for there are still two ways left to proceed. The more refined founds itself on a deep idea of Maxwell, who in his study of the statistical properties of gases, conceived an infinite ensemble of ideal replicas of the system of actual interest which populated, in his thoughts if not in reality, the various ideally possible physical states. Maxwell then sought to identify the most probable of these states with the state which, apart from certain relatively negligible fluctuations, actually obtained. His efforts created the important and successful discipline of statistical mechanics and set a potential pattern for the study of social systems which has not yet received the attention it deserves.

The second method is much more concrete and analogical, and consequently more narrow in its assertions and less certain in its implications. It consists of finding analogues, or models, of aspects of human civilization, primarily amongst the micro-organisms and insects which run through their life cycles at rates so great that the birth, development, and death of their "societies" and the eschatology of their condition can be followed and documented during an interval brief according to the standards of change of our civilizations. But a fundamental problem always intrudes: to what extent is it permissible to generalize from the rise and fall of the fruit fly Drosophila to the rise and fall of Rome, or of humanity itself? We cannot answer this question, but we also cannot avoid the belief that one of the most pressing problems which confronts anyone concerned with the future of humanity is the determination of whether, and if so, how, human society differs from the societies of lower forms insofar as the great forces which govern growth and decay are concerned.

1.1.6 Logistic Growth

Consider, for instance, the life cycle of a population of wild type Drosophila grown in a pint bottle, as illustrated in Figure 1.4. It is clear from that figure that the population does not increase indefinitely and exponentially, but rather approaches, after some brief time, an absolute limiting value beyond which it cannot pass. It is probable that no Drosphila savant would assert that either the historical record or common sense suggest that exponential growth is the norm for Drosphila society, as it generally seems and has seemed to be for us. Yet there is a certain lawfulness in the pattern of population growth displayed in Figure 1.4, called logistic growth, whose exact form need not concern us here. Suffice it to say that by means of a formula, not more complex than that which describes exponential growth itself, the calculations shown in the rightmost column of the Table below were obtained, which show a striking agreement with the observed population.



Growth of wild type *Drosophila* population in a pint bottle. The circles are the observed census counts.

Figure 4.

Figure 1.4

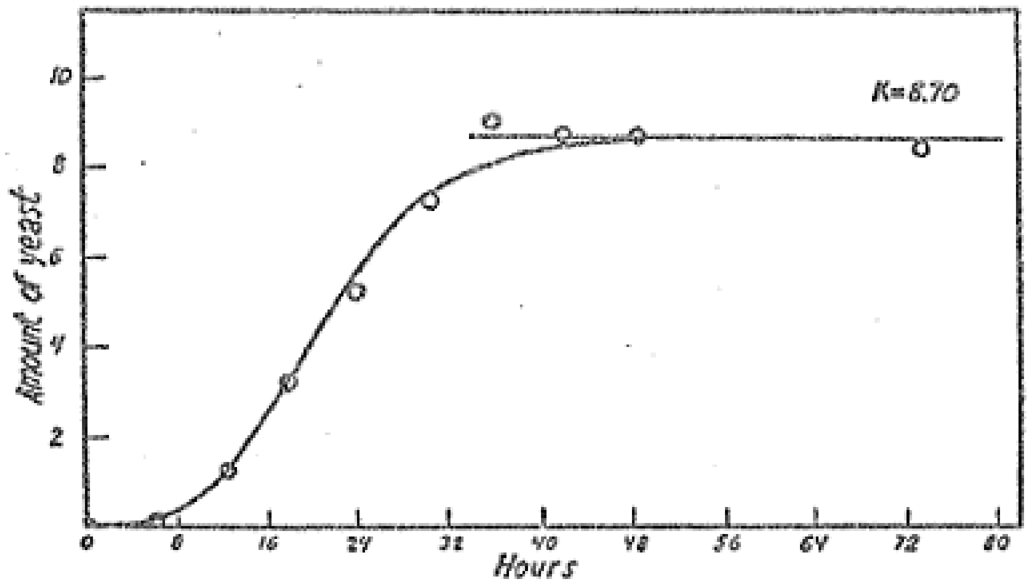


FIG. 9. Growth in volume of the yeast, *Saccharomyces cerevisiae*. From Gause ('32a).

Figure 5.

Figure 1.5

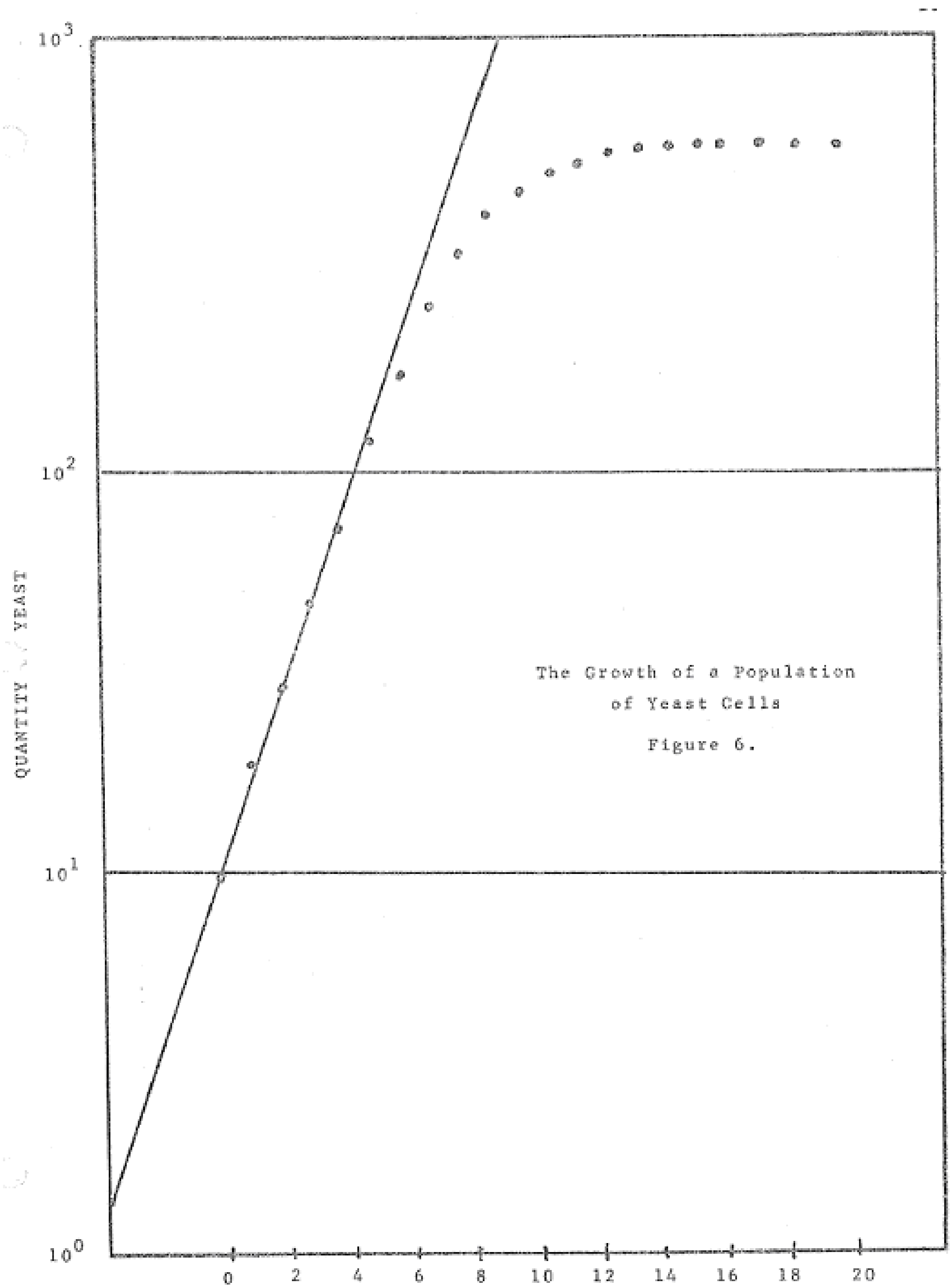
Growth of Wild Type Drosophilia Population in a Pint Bottle

Date of census	Observed population	Calculated population from equation
December 2	22	14.3
December 11	39	61.0
December 14	105	96.7
December 17	152	150.2
December 20	225	226.0
December 23	390	326.0
December 27	499	488.4
December 29	547	574.1
December 31	618	656.8
January 4	791	798.4
January 7	877	877.1
January 10	938	932.9

Table 1.1

The logistic growth pattern is as common amongst short lived rapidly reproducing lower life forms as the exponential pattern is amongst humans, and amongst people-related phenomena such as knowledge and energy inputs. Figure 5 displays the life cycle of a society of yeast cells; once again, the presence of an absolute limit beyond which population apparently cannot press is evident, and once again, the logistic mathematical description is appropriate.

In order to draw the connection between these societal microcosms which pass, from our vantage point, so quickly, through all their phases, let us reconsider the data for the yeast cell population of Figure 1.5 expressed with respect to the semi-logarithmic vertical scale such as used in Figure 1.1, and in terms of which exponential growth corresponds to straight lines. Figure 1.6, so drawn, shows that for the first 6 hours of growth, the yeast population does in fact increase exponentially, but thereafter a rapid decline in the rate of increase becomes apparent, leading after another 6 hours to a stagnant population whose numbers barely change until termination of the experiment. The reader can hardly help but notice the approximate correspondence between the early and middle periods of Figure 1.6 with the corresponding periods of exponential growth of United States population from 1650 to 1880, and the subsequent decline in the rate of increase after 1880 displayed in Figure 1.3. Are we certain that we are different from Drosophila, or from yeast cells, insofar as the cycle of population is concerned? If we are certain, on what do we base our certainty?



10³
10²
10¹
10⁰
QUANTITY YEAST
The Growth of a Population
of Yeast Cells
Figure 6.
0
2
4
6
8
10
12
14
16
18
20

Figure 1.6

Figure 1.4, Figure 1.5, and Figure 1.6 do not convey the full picture of the life cycle of a microscosmic society as it is now known, for they do not follow developments far enough into the future.

If the life cycle of the microcosmic Drosophia and yeast populations are similar to the human cycle of population and societal growth, then the former confirms our explanation of the cause of the deviation from exponentiality of the population growth of the United States) shows that it is essentially inevitable, and promises analogous declines in growth rate and asymptotic approach to stable maximum states for world population and possibly for energy consumption, productivity, growth of knowledge, etc., as well.

1.1.7 Equillibrium State

It is not difficult to envision this equilibrium state and its corresponding equilibrium society as a paradise) finally freed from the pressures and problems created by incessant population growth and its derivative phenomena, and granted the option to accommodate its desires to its means in a gradual evolutionary manner. But such a society would, necessarily, differ greatly from that to which we have become accustomed, in which savings bank deposits and corporate income offer fixed annual fractional returns by some fiducial duplication of the theological miracle of the creation of substance and value from null and void. The equilibrium society apparently promised by the Drosophila and yeast civilizations will necessarily be one of decreased personal and social mobility, decreased personal opportunity, and no doubt of decreased excitement. Each of us will have different views of the desirability of such stable circumstances.

Figure 1.4, Figure 1.5, and Figure 1.6 paint, in fact, too cheerful a picture of the population life cycle of microcosmic societies, and by implication, of our own potential future, for they do not follow developments far enough into the future. They misleadingly present the impression that an ultimate stable state of maximum population is attained by gradual increase from earlier states; they carry the implication that once society has adapted to the relatively rapid and critical conversion from exponential growth, displayed, for instance, from hours 7 through 12 in Figure 1.6, a uniform and hence rather crisis-free period of unlimited duration will follow — a period perhaps bland, possibly undesirable in certain aspects, but one at least stable. Unfortunately this is not the case, for the same forces which worked to constrain and limit exponential growth, converting it into a type of growth which is subject to an absolute upper bound as displayed in Figure 1.4, Figure 1.5, and Figure 1.6, continue to work even as population closes upon the maximum value.

1.1.8 Pollution

In the microscosmic societies these forces of constraint are imposed, on the one hand, by the geometrical restraints of the finiteness of the environment, pint bottle or Petrie dish; and on the other by the related twin factors of resource depletion and non-absorption of the byproducts of metabolism, which we generally will interpret for our more complex situation as "pollution". Whereas the direct effect of the finite environment is the absolute limitation of population, the ultimate effect of resource depletion and increasing pollutant density is a gradual diminution of the maximum value of the population that the limited environment will support. When combined, these factors suggest that the life cycle figure should in its earliest stages display unconstrained exponential growth of population when the population density is small and the ability of the environment to supply necessary resources and diffuse undesirable societal byproducts is correspondingly great, Thereafter, a period should follow wherein the geometrical constraints of the finiteness of the environment enforce an absolute limit on the supportable population. These two stages are exhibited in Figure 1.4, Figure 1.5, and Figure 1.6, and the cycle of United States population growth displays the first and the early effects of the second (Figure 1.3). A subsequent third stage follows, wherein the maximum supportable population declines gradually and steadily, ultimately to zero, so that the entire life cycle might appear somewhat as shown in Figure 1.7 below.

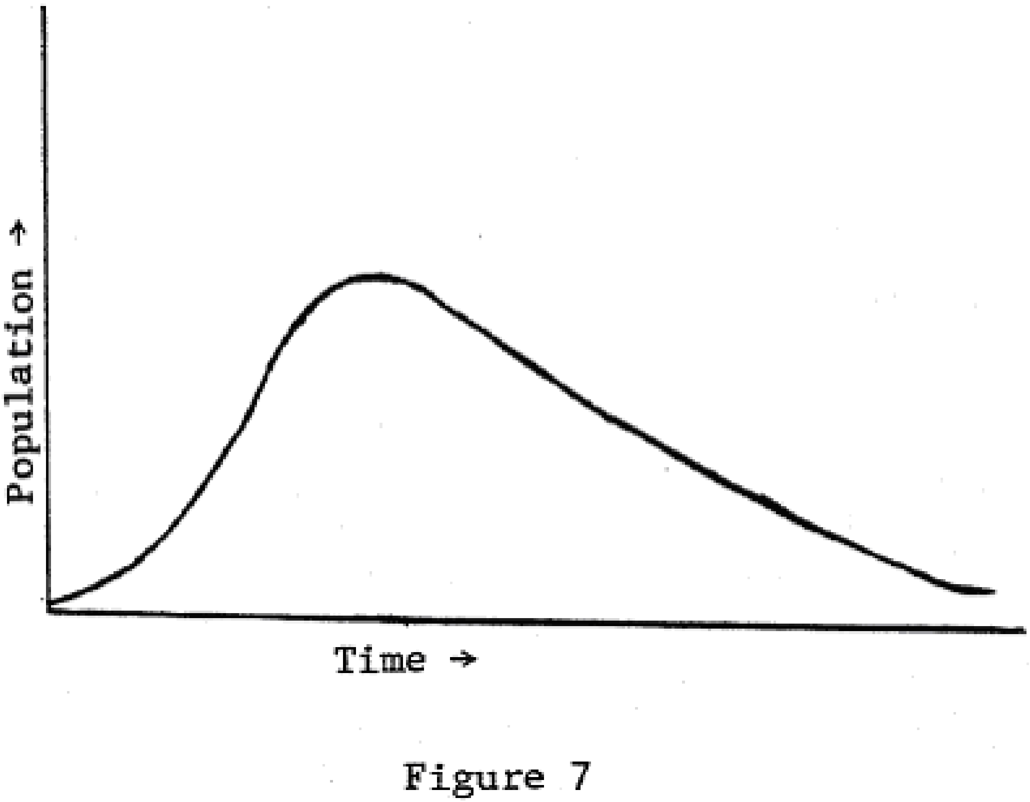


Figure 1.7

1.1.9 Time-lags

There is yet another factor which must be recognized in our description of the future population. We know that the modification of social attitudes or the realization of any great enterprise requires a certain lead time; between the decision and the effect there often intervene many years. Such time lags also occur in natural phenomena and have the utmost significance for the questions that concern us here. We may decide today to ban the use of pesticides, but the maximum value of pesticide contamination of, let us say, fish, will nevertheless not be realized for many years; we may decide, or be constrained, to stabilize population now, but population will nevertheless continue to increase for some time into the future due: to actions and decisions taken earlier but whose consequence have not yet unfolded. Even so apparently simple a matter as the national reduction of speed limits requires a not inconsiderable time interval between the impulse of necessity and reaction of implementation. So too it is amongst microcosmic societies. The natural "velocity" of population growth may carry population to magnitudes greater than those sustainable in equilibrium conditions (just as a ball thrown upward against the restraining force of gravity continues to rise for some time despite the downward tug), thereby setting the stage for subsequent decline which itself may carry population below sustainable levels. Thus we come to anticipate the possibility and indeed the probability of cyclical oscillations in the population life cycle curve, oscillations superimposed upon the general long term decline which itself follows the initial surge of exponential growth and logistic constraint. The early portions of such a curve are shown in Figure 1.8, which illustrates the life cycle of a population of Paramecia grown in a limited environment. During the first three days, the initially small Paramecia population increases exponentially; at the end of that time, the constraints of their limited environment become significant and the rate of increase of population declines to zero, while the population itself attains its maximum value at the end of 6 days. Thereafter, -it declines, at first rapidly, and then, as its density decreases, more slowly, until a local minimum value is attained at about 16 days, after which another period of increase is observed, which slows to another but this time lesser maximum, and is followed by a decline initiating a new cycle.

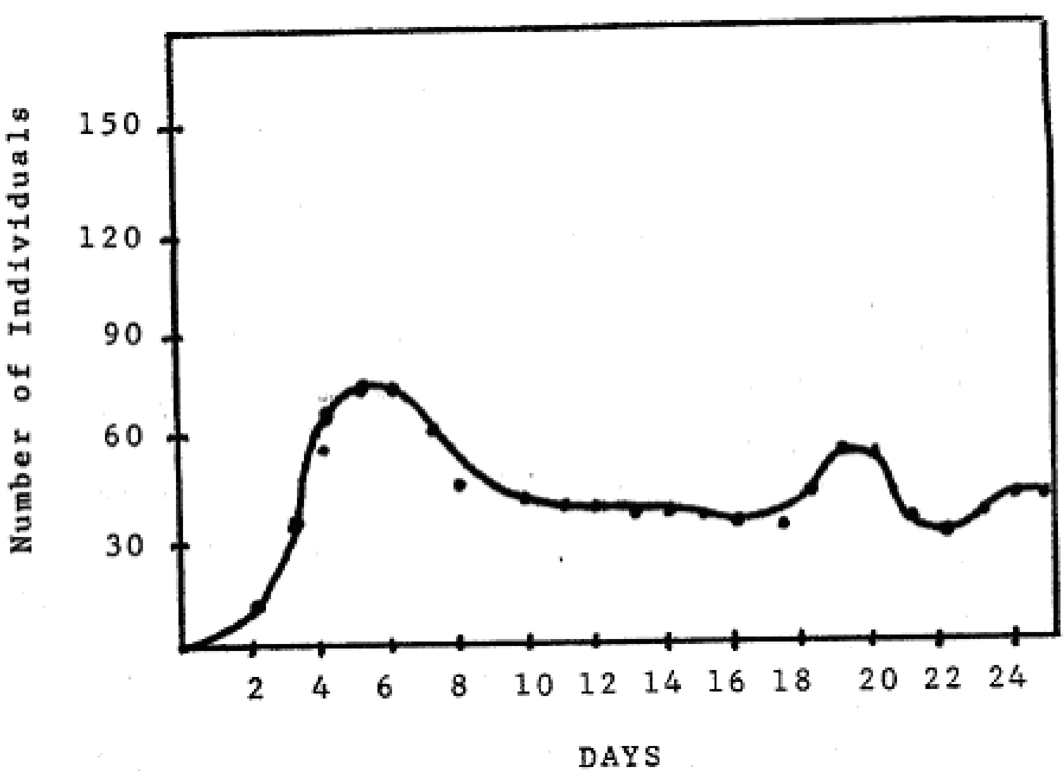


Figure 1.8

The period from 6 to 8 days constitutes an era of catastrophe for the Paramecia: population collapses to about 60 percent of its maximum value within a relatively brief interval. We can imagine governments crumbling, learning and art extinguished, a mean, brief and ugly life the reward for those who survive. By contrast, the long stable interval from 8 to 17 days which follows must appear most agreeable by contrast. One can hardly avoid drawing the parallel with the Fall of Rome and the subsequent stable medieval period.

1.1.10 Conclusion

Let us summarize the conclusions we wish to draw from the preceding remarks. First, the pattern of unconstrained exponential growth of human civilization so often found in the past is similar to the pattern of exponential growth exhibited by populations of micro-organisms and insects in their earliest phases when population densities are low. Second, the constraints imposed on the growth of the microcosmic societies by the limitations of their environment entail an absolute upper bound on their population, and similar constraints appear to apply to the various components, including population, of human societies. Both of these phases can be accurately described by simple mathematical formulae, independent of whether human or microscosmic societies are considered. Third, the microcosmic societies display an additional phase of population oscillation and ultimate decline.

We must now inquire whether this third phase may also be descriptive of the future life cycle of human population, and also whether it too can be described by mathematical formulae which lay bare its causation. If the answer to this last question is affirmative, then we will have a powerful tool with which to study the former problem.

The efforts of numerous scientific investigators have shown how this problem, at least in its gross characteristics, can be approached. One of the earliest and most distinguished of them, a truly original mind, was Vito Volterra, Professor of Mathematical Physics and Celestial Mechanics at the University of Rome, deliverer of an inaugural lecture at the founding of the Rice Institute in 1915. His theory of the "struggle for existence" prepared the foundation for all future efforts to construct a mathematical description of the interactions which determine the increase and decline of species and societies which compete with each other and amongst themselves for the limited sources of sustenance in their environment. His work, a far reaching extension of the Malthusian ideas, can be recognized in the most recent and vital computerized dynamic simulations of the world system associated with Jay Forrester, Dennis and Donella Meadows, and other contemporary scholars. The mathematical constituents of models of the Volterra and Forrester type are the formulae which describe unconstrained exponential and limited logistic growth. They are combined to reflect the structures of the various fundamental component sectors of civilization (including Population, Natural Resources, Capital Investment, and Pollution) and their intricate interactions. The resulting "life cycles" display the typical three stages exhibited by the life cycle of Paramecia (Figure 1.8 above), including the third oscillatory stage. Figure 1.9 shows the situation for the well known World Dynamics models of Forrester[12] and Meadows et. al.[11], based on the assumption that the interactive processes which are currently operative in our civilization will continue to follow their basic patterns subject only to the constraints which are naturally imposed by their mutual interaction. Population, Capital Investment, and Capital Investment in Agriculture Fraction (the amount of capital invested in agriculture) all exhibit: (the pattern of early exponential growth, logistic approach to a maximum, and the initiation of the subsequent oscillatory period.

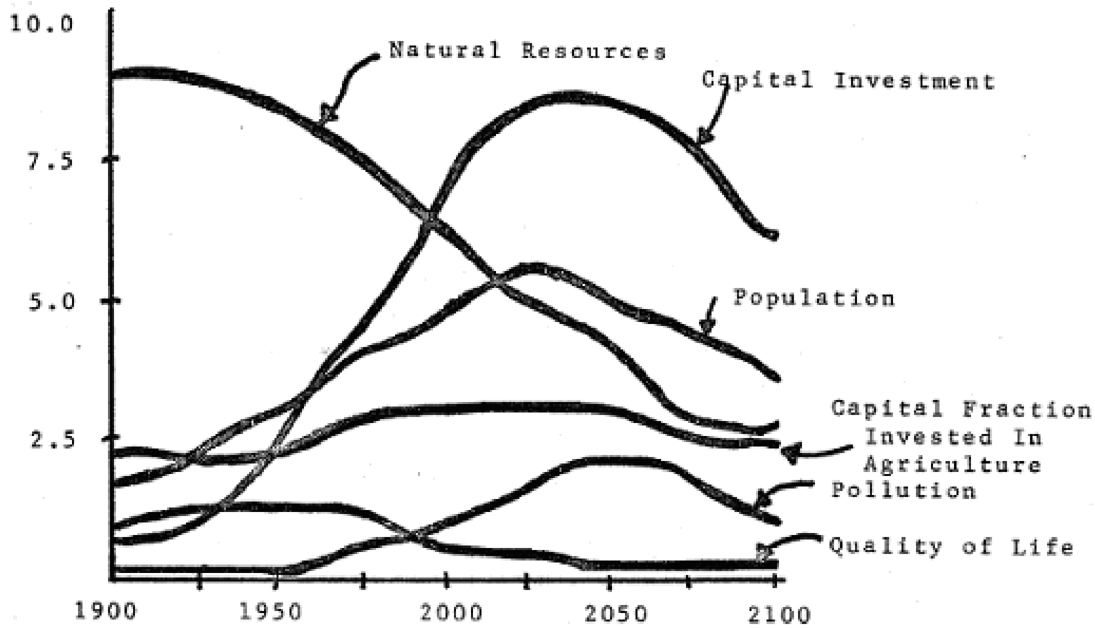
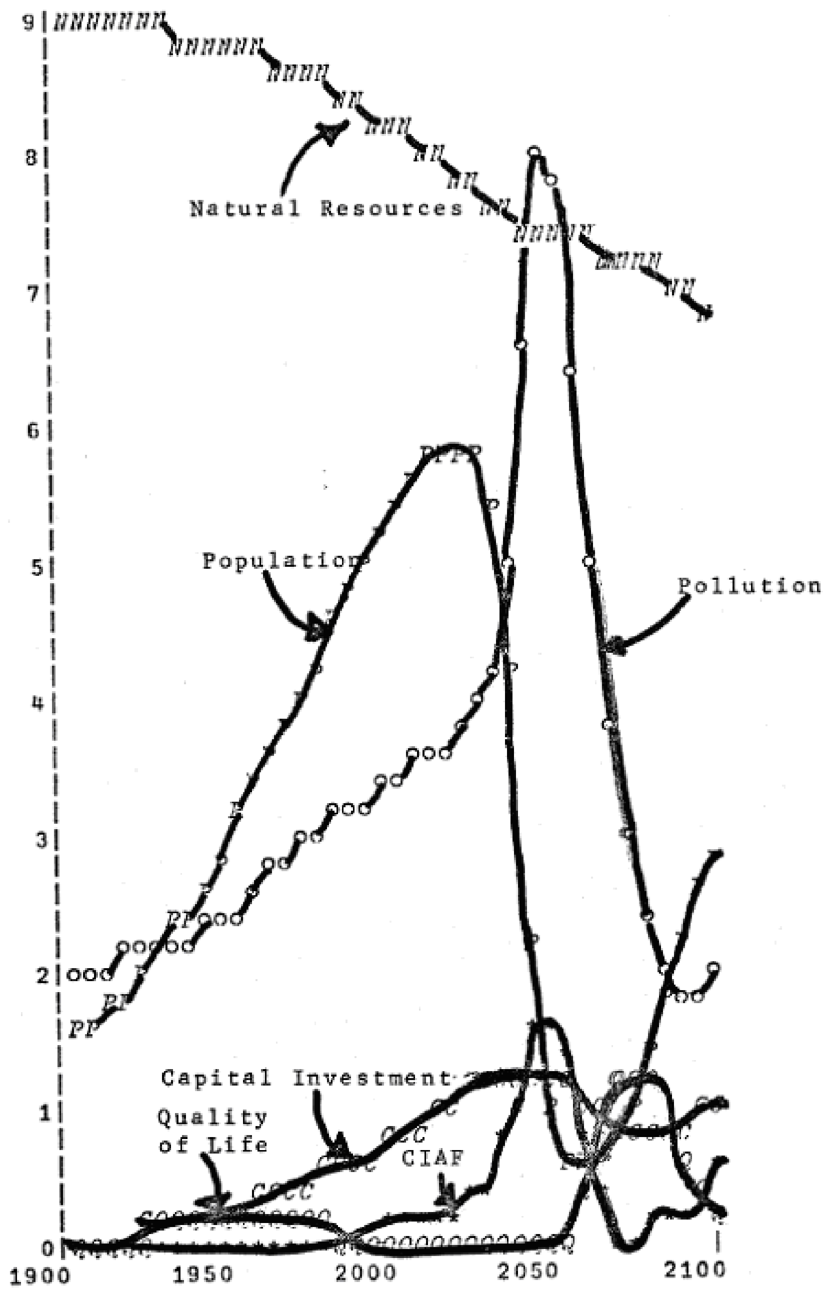


Figure 1.9: Basic Behavior of the World Model Showing the Mode in which Industrialization and Population are Surpressed by Falling Natural Resources.



Reduced Usage of Natural Resources Leads to a Pollution Crisis.

Figure 10.

Figure 1.10

The potentially catastrophic effect of the third, oscillatory, stage of development is strikingly illustrated in Figure 1.10 which displays the possible consequences of the more efficient utilization of natural resources without corresponding adjustments in the other basic sectors of civilization. Without diverting our attention to argue the merits or reliability of this particular projection, let us note the beginning of the second oscillation in each of the curves describing the life cycle of Population, Pollution, Capital Investment, and Capital Investment Fraction in Agriculture (labeled CIAF in the Figure). Were the figure drawn to another scale, the similarity to the life cycle of Paramecia in Figure 1.8 would be greatly enhanced.

We believe that the similarities between human and microcosmic societies which have been suggested above are more than superficial analogies. They justify, in our opinion, the most diligent and comprehensive investigation of the extent to which we can scientifically describe the condition of civilization and its variation with time. We must study the range of alternatives, one of which may be our future; and discover the options that are open to us for directing our destiny, insofar as it is possible, to the fulfillment of the aspirations and ultimate attainments of civilization.

Chapter 2

Models[1]

2.1 Models

The concept of a model is very close to that of an analogy. It is so fundamental to our thought, decision making, and problem solving processes that it is difficult to isolate and study; however, to accomplish the goal of simulating and understanding complicated social systems, it is essential that we do understand the concept of a model and develop methods for constructing them. Like many basic concepts, it is best described by examples:

2.1.1 Physical Model

To the average person the word "model" might bring to mind someone modeling a dress, or perhaps a photographer's model, or even a model airplane or car. Let's consider these cases. The usual reason for looking at a dress on a model is to imagine what the dress would look like on one's self without having to buy it to find out. The photographer uses his camera to make a likeness of the model in the form of a picture. The model plane or car allows one to enjoy the details and perspectives of the model without the problems and expense of actual ownership of the airplane or car. Indeed, one can try experiments on the models that would be very difficult or expensive to actually perform on the real thing.

Consider the methods and purposes behind the architect's model of a building, the car designer's prototype of a new design, or the aerodynamicist's wind tunnel model of a new airplane. As one considers what is common to these ideas of a model and what purposes are served, perhaps the concept begins to take shape.

2.1.2 Mental Model

Rather than further pursue the various types of physical models, let us consider another less obvious form of model, the mental model. For example, the merchant who mentally speculates: "if I increase the price of this article from x to y, the buyers will still buy enough that I will come out ahead," and the mother who says, "if I spank my child for leaving his toys out, he will stop", are both using intuitive mental models of incredibly complicated economic, sociological, and psychological systems that even experts don't agree on. Freud, Skinner, Erikson and others have all produced models of human psychology that much of modern therapy and advertising are based on. Indeed, reflection indicates that much of human thought is involved with mental model making and the use of these models for decision making, problem solving, or merely pleasure.

The politician who tells the voters what will happen if certain policies are followed, the advertiser who tells the potential buyer the results of using his product, and the preacher who predicts the consequences of evil to his followers are all involved with the building and use of mental and verbal models.

[1]This content is available online at <http://cnx.org/content/m17671/1.5/>.

Even the simple speculation of "if I wear these clothes I will look nice" is based on a model of how one's friends will respond to one's dress. Indeed, most processes of experience can be viewed as model building, experimentation, model modification, etc. Further reflection shows how much of one's mental activities can be viewed as involved with modeling and how many academic disciplines are based on models, even though the concept and process is poorly understood and seldom explicitly discussed.

It may seem that the idea of a model and its use is being made so general that it is useless. Our purpose in being so general is to search for what is common in these diverse examples, and to extract it for study and more efficient use.

2.1.3 Mathematical Model

Rather than follow further examples of mental or verbal models, we will turn to another form of model: the mathematical model. The incredible advances of the physical sciences and engineering disciplines have resulted from the development and use of mathematical models. When one describes the relation between the force applied to an object and its mass and acceleration by the familiar formula $F = M \cdot A$, one is using a mathematical model of a physical phenomena. Here, mathematical functions are used to represent physical qualities, so that the interrelationship can be described by equations. If these equations are fundamental and if their solutions accurately simulate the actual phenomena, then they are given the special status of "laws". Consider the cases of Newton's laws, Kirchoff's laws, Faraday's law, Boyle's law, and in other fields, Fechner's law (psychophysics), Paneto's law (economics), etc. Indeed, model formulation and verification is the basis of the so-called scientific method.

An important feature of the various types of models we have discussed is how one can move from one to another. Consider the following hypothetical account of how a physical "law" was developed.

At some point in time it was noticed that if a heavy object was dropped, it fell down. As further experience was accumulated, it was noticed that the object always fell in a straight line toward the earth. Next, after closer observation, it was discovered that the object's speed increased as it fell. The next step was a major one. It required curiosity, mathematical ability, and a real quantum jump to move from the verbal model to a mathematical one where it was conjectured that the velocity was a linear function of time after being dropped, $\mathrm{V} = \mathrm{Kt}$. This proved to be incredibly accurate, and thus, a "law" was discovered.

The use of mathematical models has been so successful in many areas that the concept of a model was sometimes forgotten. Indeed, some models are so accurate that users can forget that they are dealing with models and not the actual phenomenon. When we work in areas where accurate models are not available, a good understanding of the modeling process becomes essential.

The first step in choosing a model is deciding what the purpose of the model will be and what questions are being asked. It is obviously an advantage to use the simplest model possible to serve a particular purpose but the danger that over-simplification will destroy the validity of the model always exists.

The second step is the actual construction of the model. Here, the various theories, laws, relations, etc. that apply must be used, and after that, the model requires that new relations be established. In other words, while building the model, one often discovers what data should be collected and what experiments must be performed, as well as what data is irrelevant or misleading. At this point, alterations are often substantial in the model.

The third step is verification or validation of the model. This usually involves some comparisons of the model with the phenomenon it models. One must be very careful at this point to test all of the characteristics the model should have, while remaining within the original goals and purposes set, not violating the assumptions that were made. A common mistake is to use models outside the area for which they were intended.

Verification often involves applying the model to data that was not used in its construction to see if it can explain the observations. If internal relations were used to derive some data, these can be compared with observations. On the other hand, if the model were built by forcing agreement with the observations, then the resulting implied internal relations can be examined for their validity.

All of these steps are done in a rather circular fashion with the attempted use of, and verification of, the model suggesting modification, restructuring and reverification, or in some cases, discarding the whole approach. Some reflection will perhaps show that these are common ideas in modeling, and we try to systematically apply them to the very interesting but very difficult problem of modeling large groups of people.

Chapter 3

Dynamics[1]

3.1 Dynamics of Systems

In this module we will present several definitions and a language that will later be used to model social systems. Although a complete and detailed presentation will not be made, the ideas covered are very important for anything other than a superficial understanding of dynamic models. Much of this material grew out of what is called system theory and control theory. [9][16]

3.1.1 Definitions

As we noted before, it is sometimes difficult to give clear, precise definitions of some ideas. That is the case for the definition of a system which sounds a bit vague but seems to be as good as possible.

A system is defined as a set of interrelated entities, variables, or ideas that have some common features or purpose.

Examples of systems would be a car, a radio, a transportation network, a set of coupled equations, a society, a family, etc. The system may be physical, biological, social, conceptual, or many other forms.

The dynamics of a system is the way the various variables of the system change and evolve with time.

The study of dynamics is an important part of physics, engineering, and economics. Indeed, the study of change in history, psychology, etc. can be viewed as a study of dynamics, and when anyone makes predictions about the future, he is certainly using a dynamic model whether he realizes it or not. There are many studies of systems which are not dynamic models - these use static or equilibrium models and study relationships where time variations are assumed not important. The mathematics often used in the study of system dynamics are calculus and differential equations.

The structure of a system is the specification of the components of importance and interest and the description of the relations and interconnections within the system.

The choice of structures may be easy or very difficult depending on the system. In many physical systems the structure is fairly well developed, however, for social systems it is more complicated. The choice of age groupings, economic groupings, etc. by a sociologist is the choice of structure for a particular system. Indeed, much of the research in the social sciences has centered around structure with relatively little work being done explicitly on dynamics. For our purposes, we need both.

3.1.2 Descriptions of Systems

There are two rather different but complimentary descriptions that have been used with success in systems analysis. One is an input-output or external approach, and the other is a state variable or interval approach. Both have merits and will be briefly described.

[1]This content is available online at <http://cnx.org/content/m17819/1.4/>.

3.1.2.1 The Input-Output Description

Here there are three entities considered: the input x , output y , and the system s . Symbolically, this is illustrated by

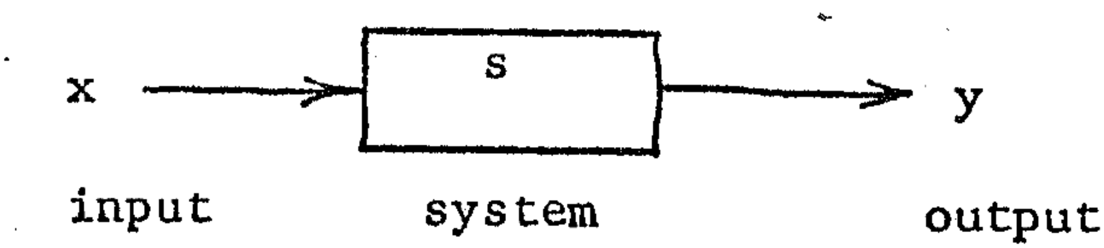


Figure 3.1

This has proven a very valuable approach that avoids internal details of the system that are of no interest or are difficult to describe.

There are three problems that can be formulated with this description:

1. Analysis: x and s given, find y ;
2. Synthesis: x and y given, find s ; and
3. Control: s and y given, find x .

In the modeling of systems and signals, one often has a partial description of all three, and they must be completed in a way to be consistent.

3.1.2.2 The State Variable or Internal

In this case, a detailed description of the internal structure of a model is made. The idea of a "state" is very important to dynamic systems, but is so fundamental as to be difficult to define. The situation if further complicated by the fact that the word state is used in many different ways in other areas.

The state of system is the present information about the past that allows one of predict the effect of the past on the future. The variables that describe the state are called the state variables and the minimum number of state variables is called the order (or dimension) of the system.

For example, if one is modeling a social system, in order to predict the future population, in addition to other factors, one must know the present population; therefore, population would be a state variable. Another example might be a second-order differential equation.

$x'' + a\dot{x} + bx = 0$

Here $x(o)$ and $\dot{x}(o)$ are needed to calculate $x(t)$; therefore, they could be stat & variables. A mechanical example would be a moving mass where one would have to know the position x and velocity v at some time to predict its future position.

In addition to state variables, a system often has many variables that are derived from present values of other variables, but do not require any past values. These are very important in the description of some systems, and it is often very difficult to distinguish between state and derived variables when initially trying to set up a model for a complex system.

The difficulty in choosing state variables is further compounded by the fact that they are not unique. (Their number is, however.) For example, in a system of equations, a change of variables could be carried out and the new variables used as states. In the mechanical example, one could choose $v + x$ for one state variable and $\frac{1}{2}v - \sqrt{2x}$ for the other, although it's hard to imagine why one would want to.

3.1.3 Deterministic and Probabilistic

Still another division of description is into those that use deterministic equations to relate the various system variables and those that relate the statistics of the variables. These two approaches are complimentary. For example, in describing a gas in a container, one can relate the gross characteristics of pressure, temperature and volume by an algebraic equation; however, one must resort to statistics to describe an individual molecule. In the case of the social model, it seems to also hold that individual people or small groups must be described statistically, but the gross behavior of large aggregates can be described deterministically. This is certainly not as clear-cut as for a container of gas, but it is what we will follow.

Indeed, not only is the decision between a deterministic and probabilistic model difficult to make for a social system, but the choice of structure, state variables, and many other factors are all difficult and the subject of much debate a long researchers. What this means, however, is the basic concepts and definitions must be understood even better and used with even greater care.

3.1.4 Classifications

There are a number of rather common classifications of systems that prove useful. The two most important are given here in terms of an input-output description.

A. A system is called linear if, and only if, the following two conditions hold. In an input x_1 causes an output y_1, and an input x_2 causes an output y_2, then an input which is the sum of two inputs, $x_1 + x_2$, must cause an output $y_1 + y_2$. This is called superposition. If the input x_1 is scaled by an arbitrary value a , then the resulting output must also be scaled by the same value a .

$$F = M \cdot A \tag{3.1}$$

$$\text{If} \quad x_1 \to y_1 \quad \text{and} \quad x_2 \to y_2 \tag{3.2}$$

$$\text{then} \quad (x_1 + x_2) \quad \to \quad (x_1 + y_2) \tag{3.3}$$

$$\text{and} \quad ax_1 \quad \to \quad ay_1 \tag{3.4}$$

B. A system is called time-invariant or stationary if, and only if, the following is true for arbitrary t .

If $\quad x(t) \to y(t) \quad$ then $\quad x(t+T) \to y(t+T)$.

3.1.5 Feedback

A particular structure [Luenberger 1979] which is so important that it warrants special discussion has the feature that the output affects the input. This is illustrated by the following figure.

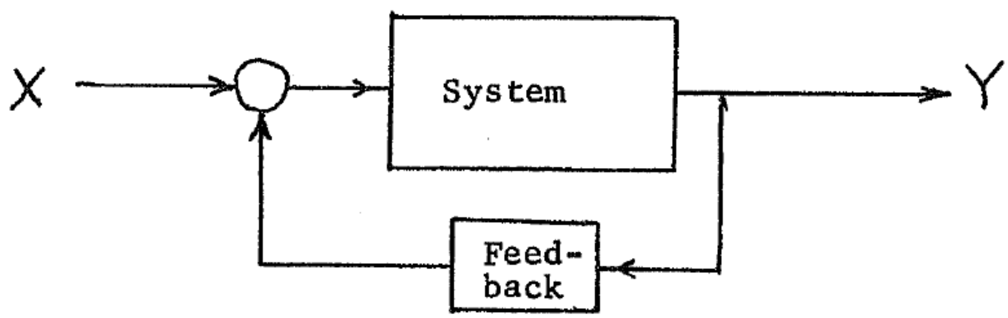


Figure 3.2

Feedback is often part of naturally occurring systems and it also is often a part of constructed systems. The most common feedback system is probably the thermostat that uses a measured temperature to feedback a controlling signal to a heater in an oven or room heating system. The filling mechanism in the tank of a toilet uses a float to feedback a measure of the water level to control the input valve. A person's blood sugar level is controlled by a complicated biological feedback system. The power steering of a car, the auto-pilot of an airplane, and the control of a satellite rocket are all examples of feedback.

An interesting model using feedback can be used to describe a bank savings account. Here the output can be the amount of interest earned which is then fed back and added to increase the account. This feedback is called compounding, and results in the rapid exponential growth of an account.

A similar model will be used to describe a population where the feedback signal is the number of people added by births less the number of deaths. This forms the basis of the exponential predictions of population growth, and we will explore it in detail later.

The basis of the free marketplace is based on feedback through price changes to cause the supply to follow the demand.

While feedback is a useful concept, its effects become more difficult to predict as the systems become more complex. A simple example illustrates one problem. Consider a person adjusting the temperature of his shower by the hot and cold valves. If there is a time delay introduced by a length of pipe between the valves and the shower head, the person will over control. If the water is initially too hot, he will turn on the cold water, but because of the delay, no effect is immediately felt so more cold water is turned on. This continues until finally the now very cold water reaches the shower head, whereupon the person starts the same procedure of increasing the hot water. This oscillation will continue until the person "gets smart" and allows for the delay. A similar problem can occur in college education because of the four-year delay between the choice of a major and the graduation to a job.

A bit of reflection begins to show the complicated nature of a social system will involve multi-variable nonlinear systems with time delays and multiple feedback loops.

Chapter 4

Exponential Growth[1]

Since it is the dynamic nature of a system that we want to model and understand, the simplest form will be considered. This will involve one state variable and will give rise to so-called "exponential" growth.

4.1 Two Examples

First consider a mathematical model of a bank savings account. Assume that there is an initial deposit but after that, no deposits or withdrawals. The bank has an interest rate r_i and service charge rate r_s that are used to calculate the interest and service charge once each time period. If the net income (interest less service charge) is re-invested each time, and each time period is denoted by the integer n, the future amount of money could be calculated from

$$M(n+1) = M(n) + r_i M(n) - r_s M(n) \tag{4.1}$$

A net growth rate r is defined as the difference

$$r = r_i - r_s \tag{4.2}$$

and this is further combined to define R by

$$R = (r+1) \tag{4.3}$$

The basic model in (4.1) simplifies to give

$$M(n+1) = (1 + r_i - r_s)\ M(n) \tag{4.4}$$

$$= (1+r)\ M(n) \tag{4.5}$$

$$M(n+1) = R\ M(n)\ . \tag{4.6}$$

This equation is called a **first-order difference equation**, and the solution $M(n)$ is found in a fairly straightforward way. Consider the equation for the first few values of $n = 0, 1, 2, ...$

$$M(1) = R\ M(0) \tag{4.7}$$

$$M(2) = R\ M(1) = R^2\ M(0) \tag{4.8}$$

[1]This content is available online at <http://cnx.org/content/m17672/1.3/>.

$$M(3) = R\ M(2) = R^3\ M(0) \tag{4.9}$$

$$\cdot\ \cdot\ \cdot \tag{4.10}$$

$$M(n) = M(0)\ R^n \tag{4.11}$$

The solution to (4.4) is a geometric sequence that has an initial value of $M(0)$ and increases as a function of n if R is greater than 1 (r 0), and decreases toward zero as a function of n if R is less than 1 (r 0) . This makes intuitive sense. One's account grows rapidly with a high interest rate and low service charge rate, and would decrease toward zero if the service charges exceeded the interest.

A second example involves the growth of a population that has no constraints. If we assume that the population is a continuous function of time $p(t)$, and that the birth rate r_b and death rate r_d are constants (not functions of the population $p(t)$ or time t), then the rate of increase in population can be written

$$\frac{dp}{dt} = (r_b - r_d)\ p \tag{4.12}$$

There are a number of assumptions behind this simple model, but we delay those considerations until later and examine the nature of the solution of this simple model. First, we define a net rate of growth

$$r = r_b - r_d \tag{4.13}$$

which gives

$$\frac{dp}{dt} = rp \tag{4.14}$$

which is a first-order linear differential equation. If the value of the population at time equals zero is p_o , then the solution of (4.14) is given by

$$p(t) = p_o\ e^{rt} \qquad p_o = p(0) \tag{4.15}$$

The population grows exponentially if r is positive (if r_b r_d) and decays exponentially if r is negative (r_b r_d). The fact that (4.15) is a solution of (4.14) is easily verified by substitution. Note that in order to calculate future values of population, the result of the past as given by $p(0)$ must be known. ($p(t)$ is a state variable and only one is necessary.)

4.2 Exponential and Geometric Growth

It is worth spending a bit of time considering the nature of the solution of the difference (4.4) and the differential (4.14). First, note that the solutions of both increase at the same "rate". If we sample the population function $p(t)$ at intervals of T time units, a geometric number sequence results. Let p_n be the samples of $p(t)$ given by

$$p_n = p(nT) \qquad n = 0,\ 1,\ 2,\ ... \tag{4.16}$$

This give for (4.15)

$$p_n = p(nT) = p_{o\,e}^{rnT} = p_o\left(e^{rT}\right)^n \tag{4.17}$$

which is the same as if

$$R = e^{rT} \tag{4.18}$$

This means that one can calculate samples of the exponential solution of differential equations exactly by solving the difference (4.4) if R is chosen by (4.18). Since difference equations are easily implemented on a digital computer, this is an important result; unfortunately, however, it is exact only if the equations are linear. Note that if the time interval T is small, then the first two terms of the Taylor's series give

$$R = e^{rT} \approx 1 + rT \tag{4.19}$$

which is somewhat similar to (4.3). Another view of the relation can be seen by approximating (4.14) by

$$\frac{x(n+1) - x(n)}{T} = r\,x(n) \quad , \tag{4.20}$$

which gives

$$x(n+1) = x(n) + rT\,x(n) \tag{4.21}$$

$$= (1 + rT)\;x(n) \tag{4.22}$$

having a solution

$$x(n) = x(0)\;(1 + rT)^n \tag{4.23}$$

This implies (4.21) also, and the method is known as Euler's method for numerically solving a differential equation.

These approximations are used often in modeling. For population models a differential evuation is often used, even though it is obvious that births and deaths occur at random discrete times and populations can take on only integer values. The approximation makes sense only if we use large aggregates of individuals. We end up modeling a process that occurs at random discrete points in time by a continuous time mode, which is then approximated by a uniformly-spaced discrete time difference equation for solution on a digital computer!

The rapidity of increase of an exponential is usually surprising and it is this fact that makes understanding it important. There are several ways to describe the rate of growth.

$$\text{If} \quad x = k\,e^{rt} \quad , \tag{4.24}$$

$$\text{then} \quad \frac{dx}{dt} = k\;r\;e^{rt} \tag{4.25}$$

$$\text{or} \quad r = \frac{1}{x}\,\frac{dx}{dt} \quad . \tag{4.26}$$

This states that r is the rate of growth per unit of x. For example, the growth rate for the U.S. is about 0.014 per year, or an increase of 14 people per thousand people each year.

Another measure of the rate is the time for the variable to double in value. This doubling time, T_d, is constant and can easily be shown to be given by

$$T_d = \frac{1}{r}\;log_e 2 = 0.6931472\;\frac{1}{r} \tag{4.27}$$

For example, doubling times for several rates are given by

r	T_d
.01	70
.02	35
.03	23
.04	17
.05	14
.06	12

Table 4.1

The present world population is about three billion, and the growth rate is 2.1% per year. This gives

$$p(t) = 3\ e^{0.021\ t} \tag{4.28}$$

with $p(t)$ measured in billions of people and t in years. This gives a doubling time of 33 years. While it is easy to talk of growth rate and doubling times, these have real predictive meaning **only** if the growth is exponential.

4.3 Two Points of View

There are two rather different approaches that can be used when describing some physical phenomenon by exponential growth. It can be viewed as an empirical description of how some variables tend to evolve in time. This is a data-fitting view that is pragmatic and flexible, but does not give much insight or direction on how to conduct experiments or what other things might be implied.

The second approach primarily considers the underlying differential equation as a "law" of growth that results in exponential behavior. This law has various assumptions and implications that can be examined for reasonableness or verified by independent experiment. While perhaps not so important for the first-order linear equation here, this approach becomes necessary for the more complicated models later.

These approaches must often be mixed. The data will imply a model or equation which will give direction as to what data should be taken, which will in turn imply modifications, etc. The process where structure is chosen and the parameters are chosen so that the model solution agrees with observed data is a form of parameter identification. That was how (4.28) was determined.

4.4 The Use of Semi-Log Plots

When examining data that has been plotted in fashion, it is often hard to say much about its basic nature. For example if a time series is plotted on linear coordinates as follows

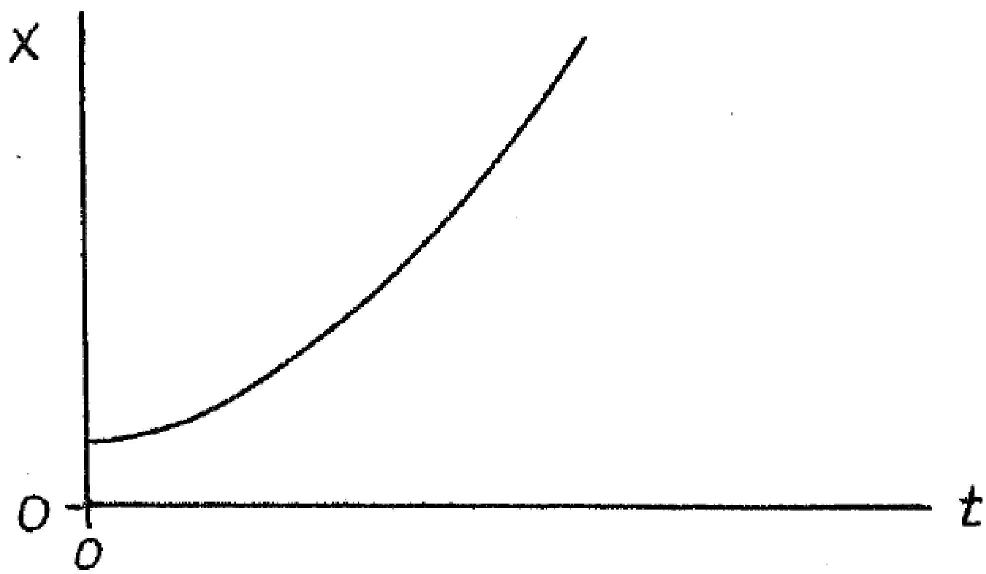


Figure 4.1

it would not be obvious if it were samples of an exponential, a parabola, or some other function. Straight lines, on the other hand, are easy to identify and so we will seek a method of displaying data that will use straight lines.

If $x(t)$ is an exponential, then

$$x(t) = k\ e^{rt} \tag{4.29}$$

Taking logarithms of base e for both sides of (4.29) gives

$$log\ x = log\ k + rt \tag{4.30}$$

If, rather than plotting x versus t, we plot the log of x versus t, then we have a straight line with a slope of r and an intercept of log k. It would look like

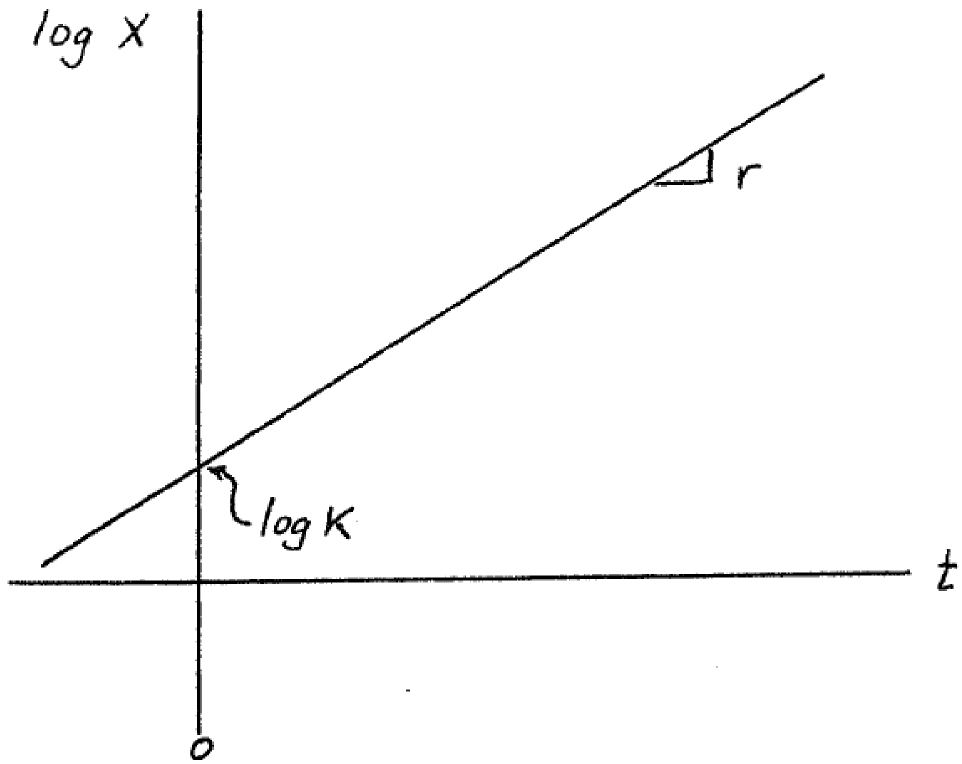


Figure 4.2

Actually using the logarithm of a variable is awkward so the variable itself can be plotted on logarithmic coordinates to give the same result. Graph paper with logarithmic spacing along one coordinate and uniform spacing along the other is called semi-log paper.

Consider the plot of the U.S. population displayed on semi-log paper in Figure 3. Note that there were two distinct periods of exponential growth, one from 1600 to 1650, and another from 1650 to 1870. To calculate the growth rate over the 1650 – 1870 period, we can calculate the slope.

$$r = \frac{\log\ p(t_1) - \log\ p(t_2)}{t_1 - t_2} \tag{4.31}$$

$$= \frac{log\ 10^7 \quad - \quad log\ 10^5}{1823 \quad - \quad 1670} \tag{4.32}$$

$$= \frac{16.118 \quad - \quad 11.513}{153} = 0.03 \tag{4.33}$$

During that period, there was a 3% per year growth rate or, in other terms, a 23-year doubling time.

An alternative is to measure the doubling time and calculate r from (4.27). Still another approach is to measure the time necessary for the population to increase by $e = 2.72...$ The growth rate is the reciprocal of that time interval. Derive and check these for yourself.

The data displayed in Figures 1, 2 and 3 illustrates exponential growth and the use of semi-log paper. The books [7] [17] give interesting discussions of growth.

4.5 Analytical versus Numerical Solutions

In some cases, there is a choice between an analytical solution in the form of an equation and a numerical solution in the form of a sequence of numbers. A real advantage of an analytical form is the ability to easily see the effects of various parameters. For example, the exponential solution of (4.12) given in (4.15) directly shows the relation of the growth rate r in equation (4.12) to the exponent in solution (4.15). If the equation were numerically solved, say on a digital computer or calculator using Euler's method given in (4.21) , it would take numerous experimental runs to establish the same relations.

On the other hand, for complicated equations there are no known analytical expressions for the solutions, and numerical solutions are the only alternatives. It is still worth studying the analytical solution of simple equations to gain insight into the nature of the numerical solutions of complex equations.

4.6 Assumptions

The linear first-order differential equation model that is implied by exponential growth has many assumptions that are worth noting here. First, the growth rate is constant, independent of crowding, food, availability, etc. It also assumes that age distribution within the population is constant, and that an average birth and death rate makes sense. There are many factors one will want to include effects of crowding and resource availability, time delays in reproduction, different birth and death rates for different age groups, and many more. In the next section we will add one complication the effects of a limit to growth.

Chapter 5

Limits to Growth[1]

5.1 A Limit to Simple Growth: The Logistic Function

In the preceding sections we saw how the linear first-order differential equations lead to exponential solutions. Both the unbounded nature of the solution and the assumption of a constant growth rate indicate a modification that will give more realistic modeling of observed population growths.

A. A Nonlinear Equation

It seems reasonable to assume under that many conditions the growth rate r would not remain constant, but would decrease with increasing population. This might result from the effects of crowding and reduced resources, or other physical and psychological factors on the birth and death rates. The simpliest functional form one could assume would be a linear reduction of r as the population increased.

$$r = r_o\,(1 - \beta\,p) \tag{5.1}$$

Here r_o is the rate for very small p where no limits have been felt. The term β gives the reducing effect of p on r. This is seen by plotting r versus p.

[1]This content is available online at <http://cnx.org/content/m18163/1.3/>.

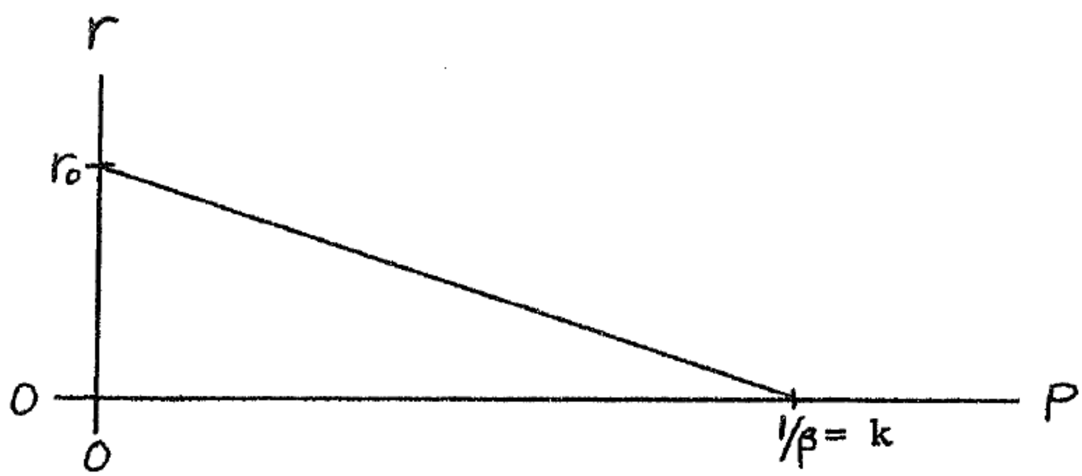


Figure 5.1

Note the growth rate is maximum at r_o for p equal to zero and linearly decreasing to zero when p is equal to $\frac{1}{\beta}$. The constant case is a special instance of (5.1) for β equal to zero. Now consider the population equation with this new growth rate.

$$\frac{dp}{dt} = r\ p \tag{5.2}$$

$$= r_{o\,(l\ -\ \beta\ p)\ p}(5.3)$$

$$\frac{dp}{dt} = r_{o\,p} - r_o\ \beta\ p^2 \tag{5.4}$$

This is now a nonlinear first-order differential equation with several interesting features. The solution of (5.2) can be shown to equal

$$p\,(t) = \frac{ke^{r_o t}}{1+\beta ke^{r_o t}} = \frac{1}{\beta + k^{-1}e^{-\mathrm{r}_o t}} \tag{5.5}$$

for

$$k = \frac{p\,(o)}{1\ -\ p\,(o)\ \beta} \tag{5.6}$$

This function is called a **logistic** or **sigmoid,** and is illustrated below for several initial values of $p\,(o)$.

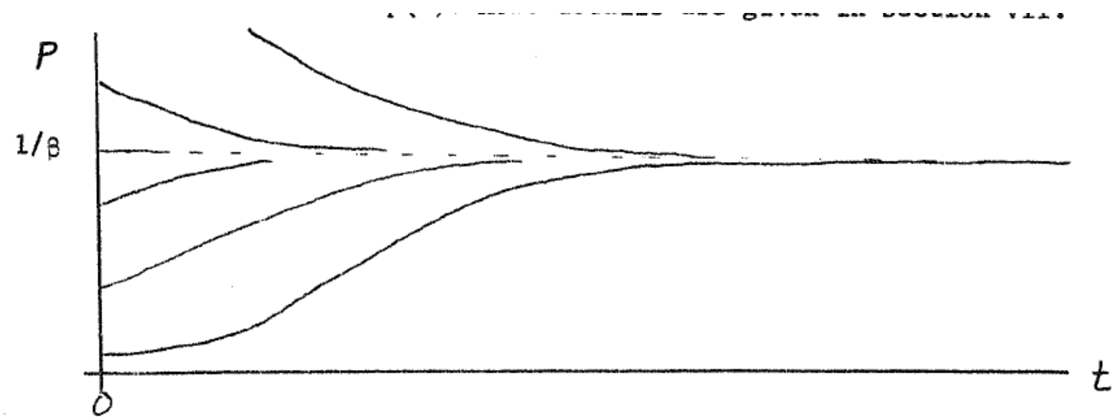


Figure 5.2

An alternate form is

$$\dot{p} = r_o \left(1 - \frac{p}{k}\right) p \quad (5.7)$$

where k is $p(\infty)$, the asymptotic value of population, or the carrying capacity of the "system". This requires $k = \frac{1}{\beta}$.

There are several very interesting features of this function. For small initial populations, the initial increase is very much like exponential. This is obvious since the negative term in (5.2) is small and the equation looks linear. However, as $p(t)$ grows, the growth rate goes to zero and a steady-state or equilibrium is approached as a limit.

It it interesting to note that it is possible to normalize the logistic into a "standard" form. If we scale both the amplitude and time by

$$x = \beta p \tau = \beta t + \log \eta k \; , \quad (5.8)$$

then (5.5) becomes

$$x(\tau) = \frac{1}{1 + e^{-\tau}} \quad (5.9)$$

When plotted on semi-log paper, the logistic is an increasing straight line for small time, and becomes a horizontal straight line for large time.

The use of the logistic to model simple population growth with a limit is shown in Figures 4, 5, and 6. There have been many other applications of the logistic [8] [18] [20] [5] with some success and some failures. Unfortunately, if one tries to use this model to predict the limiting value while the system is still in the early stages of growth, a large error results since an exponential and a logistic look very similar in the early stages.

It is possible to manipulate the data so that a plot of it becomes a straight line. If the reciprocal of $p(t)$ in (5.5) is considered

$$\frac{l}{p} = \beta + k^{-1} e^{-r o_t} \quad (5.10)$$

and the logarithm is taken

$$\log \left(\frac{l}{p} - \beta \right) = - \log k - \beta_t \quad (5.11)$$

we have a linear function. Unfortunately, trial-and-error must be used to find β since it appears on both sides of the (5.11).

5.2 A More Complicated Limit to Growth

The (5.2) that gives rise to the logistic is the simpliest modification to include the effect of a non-constant r. Perhaps a more realistic relation of r as a function of p could be found. In general, the equation becomes

$$\frac{dp}{dt} = f(p) \tag{5.12}$$

where $f(p)$ is a more complicated function of p. The nature of the solution will still be the same in the sense that $p(t)$ will move from an initial value monotonically toward a constant limit – there can be no over or under shoots with the model. There may be more than one possible final steady-state value if $f(p)$ has more than one zero.

Consider the growth rate to vary as a quadratic function of population rather than as a linear function shown in (5.1). One particular form it might take could be

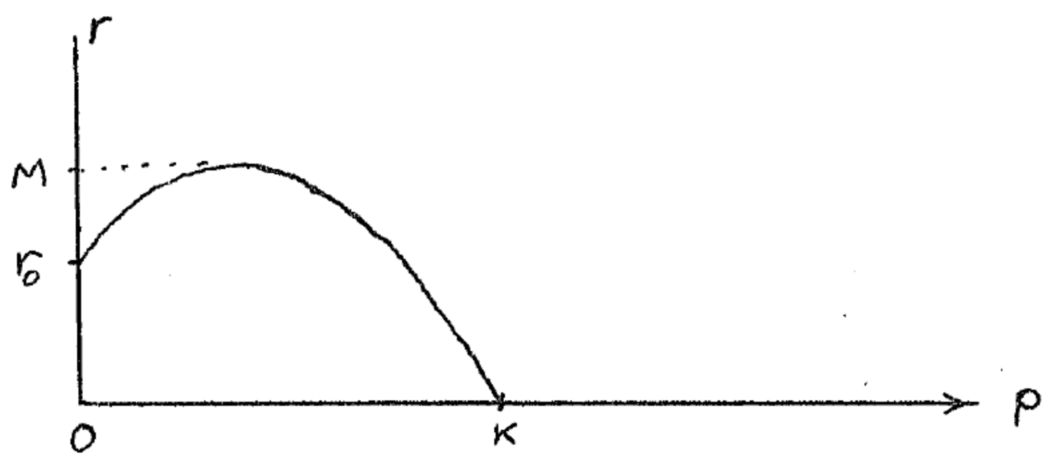


Figure 5.3

where

$$r = r_{o(1 + Bp - Cp^2)} \tag{5.13}$$

This might be the model of a system where moderate increases in population increase the growth rate, but higher values finally cause the rate to drop as before. A situation where moderate levels allow male and female members to find each other more often could lead to this model, or a process such as industrialization, where efficiency can increase with size to a point.

The resulting differential equation is first order, but now has a cubic nonlinearity.

$$\frac{dp}{dt} = r_{op} + r_o Bp^2 - r_{oCp}^3 \tag{5.14}$$

The solution looks somewhat like a logistic. It starts with low population and an exponential growth; then, as the second term begins to dominate, the growth is super-exponential – faster than exponential. Finally, the cubic term – the limit – causes an abrupt leveling off to a constant equilibrium value of k .

If more complicated systems are to be modeled, modification other than more complicated nonlinearities must be used. Other modification might include

a. Time delays
b. Use population age groups
c. Allow parameters to depend on other environmental variables or states.

The next generalization we will consider here will be the addition of another state variable. This allows a much broader and more versatile system of equations and solutions.

Chapter 6

Simulation[1]

6.1 Dynamic Simulation on a Digital Computer

If a system is modeled by a differential equation, and if the equations are numerically solved on a digital computer or calculator, the system is said to be simulated on the computer. If the model is valid and the numerical methods accurate, experiments can be performed on the computer simulation that might be impossible to conduct otherwise.

Consider several examples that use the models already discussed. If a population is governed by a linear first-order equation

$$\frac{\mathrm{dp}}{dt} = rp \tag{6.1}$$

one would not be able to "solve" this equation on a computer. If, however, we use Euler's method as was done in (14) by approximating the derivative as

$$\frac{dp}{dt} = \frac{p(nT + T) - p(nT)}{T} \tag{6.2}$$

where time is considered at intervals of T ,

$$t = nT \qquad n = 0,\ 1,\ 2,\ ... \tag{6.3}$$

This gives for (1) .EQ (34)

$$p(nT + T) = p(nT) + Trp(nT)\,p(nT + T) = (1 + rT)\ p(nT) \tag{6.4}$$

If we include the time interval T in the functional notation by

$$p(nT) = x(n) \tag{6.5}$$

then (3) becomes

$$x(n + 1) = (1 + rT)\ x(n) \tag{6.6}$$

which is now in a form that one can easily calculate successive values of $x(n)$ given any initial value. This can be programmed on a computer or simply done on a hand calculator.

Next, consider the nonlinear equation that models a population with a simple limit given by (23).

$$\frac{dp}{dt} = \left(1 - \frac{p}{k}\right) r_{o\,p} \tag{6.7}$$

[1] This content is available online at <http://cnx.org/content/m18164/1.2/>.

Using Euler's method again gives

$$x(n+1) = x(n) + Tr_{o\,x(n)} - \left(\frac{Tr_o}{k}\right) x(n)^2 \tag{6.8}$$

This equation is complex enough to illustrate several points; therefore, we will examine several numerical solutions. (6.8) was programmed on a Tektronix 31 programmable calculator with a plotter automatically plotting the solutions by drawing straight lines between successive $x(n)$.

First, consider a low-density growth rate of $r_o = 0.1$ or 10% per year for an initial population of $p_o = 100$ over a time period of 100 years. We will use for the reducing effect on the growth rate in (2), a value of $\beta = 0.0001$, which implies a carrying capacity for the system of $k = 10,000$. For the Euler method, a time interval of $T = 2$ years is used, which means 50 calculations of (6) will be necessary for the 100-year period.

The curves in Figure A are the output of the simulation for the above parameters and also for other growth rates of 5% and 20%. Note the solution always approaching the same limit but requiring different amounts of time.

In Figure B, the model is run assuming several different initial populations. Again, the solutions always approach the limit of k, even if the initial population is greater than k .

Figure C shows the effects of various amounts of limiting by considering various values for the factor β, and therefore k, the carrying capacity. When the limit is removed ($k = \infty$), the growth is exponential.

These examples illustrate the kinds of questions that can be pursued by running experiments on the computer simulation. There is one more point that should be considered. It has nothing to do with the differential equation model (23) but with the numerical procedure, Euler's method. Consider the effects of using various time intervals T while holding everything else constant. Figure D shows the results of this experiment. The curve resulting from using a time interval of $T = 0.2$ years looks essentially the same as the exact solution of the differential equation. The numerical solution deviates more as T is increased until, for $T = 20$ years, it has lost the character of the exact solution. A method for checking to see if T is sufficiently small is to try halving it until the change is small.

One last point should be made concerning this numerical simulation. Euler's method is the only approach to numerically solve (23) that has been discussed. That is not because it is the best – there are far more efficient and sophisticated methods – but that is not our subject here, so we will continue with the straightforward algorithm of Euler.

The super-exponential logistic equation of (31) was simulated on the calculator and run with a low population growth rate of $r_o = 0.1$, a maximum rate of $M = .2$, and a carrying capacity of $k = 10,000$. This solution is shown in Figure E and compared with an exponential of the same r_o , and a simple logistic of the same r_o and k. The model was run again with a maximum growth rate of $M = .5$, and the results are shown in Figure F. Note the initial exponential growth which becomes super-exponential, growing extremely rapidly, then abruptly leveling off to an equilibrium.

Chapter 7

Second Order Model[1]

7.1 Second-order or Two-state Variable Systems

In the last few sections, we discussed first-order models of various systems and studied the types of interactions that could be modeled and the nature of the solutions of these models. Of the several indicated generalizations that could be made, this section will consider adding another state variable, so that the effects of two interacting variables can be used and studied. This will greatly increase the class of systems we can model and the class of solutions that result. In addition, a very interesting set of classical problems fall into this class with interesting solutions and interpretations.

To illustrate the general problem, consider a system that contains populations of two different types with distinctly different characteristics. Assume these two populations have a strong effect on each other, as well as being influenced differently by their environment, so that modeling them by a single total population would not yield useful results. We must, therefore, have two separate state variables to describe the systems, and this could perhaps be done in the following way.

$$\frac{dp_1}{dt} = f\ (p_1,\ p_2) \tag{7.1}$$

$$\frac{dp_2}{dt} = g\ (p_1,\ p_2) \tag{7.2}$$

Here the rate of change of population p_1 is assumed to depend on both the populations p_1 and p_2 ; and likewise, the rate of change of p_2 is assumed to depend on p_1 and p_2 , but in perhaps a different way.

Many types of interactions could be considered. It might be that p_1 and p_2 compete for the same source of food or resources; it might be that p_1 is a prey of the predator p_2 ; or it could be that they both contribute to the welfare of the other. These assumptions would be implemented in the choice of $\underline{f}$ and $\underline{g}$ to describe the particular case. The best known classical models of these types were proposed by Lotka (1925) and Volterra (1926). Later, Gause (1934) did further experimental and interpretative work. Most of this type of work was done in population ecology.[19], [24].

A. The Simple Lotka-Volterra Competition Model

Consider the particular for for the two-variable model to be

$$\frac{dp_1}{dt} = a\ p_1 - b\ p_1\ p_2 \tag{7.3}$$

$$\frac{dp_2}{dt} = c\ p_2 - d\ p_1\ p_2 \tag{7.4}$$

[1] This content is available online at <http://cnx.org/content/m18165/1.2/>.

This might be a simple model of two competing populations, where a and c are the net rate of increase that would occur if the other population did not exist. The coefficients b and d model the negative effects of interaction on the rates of change as a measure of how often one encounters the other.

To simplify the mathematics, a change of variables will be made. Consider the rearrangement of (7.3) into

$$\left(\frac{d}{c}\right)\dot{p}_1 = a\left(\left(\frac{d}{c}\right)p_1 - \left(\frac{b}{a}\right)p_2\left(\frac{d}{c}\right)p_1\right) \tag{7.5}$$

$$\left(\frac{b}{a}\right)\dot{p}_2 = c\left(\left(\frac{b}{a}\right)p_2 - \left(\frac{d}{c}\right)p_1\left(\frac{b}{a}\right)p_2\right) \tag{7.6}$$

Now let $x = \left(\frac{d}{c}\right)p_1$ and $y = \left(\frac{b}{a}\right)p_2$ then, (7.5) becomes

$$\dot{x} = a(x - xy) \tag{7.7}$$

$$\dot{y} = c(y - xy) \tag{7.8}$$

Note that x and y are related to p_1 and p_2 by simple constant multipliers or scale factors, and therefore, the nature of the solution of (7.7) is the same as (7.3), but now there are only two parameters, a and c, to consider. In fact, by allowing a change of scale of the time variable, it is possible to reduce the number of parameters to one, but we will not do that. The problem of solving the coupled equation of (7.7) or, more generally, of (7.1) can be approached three ways. In some cases, an analytical equation for the solution can be found. This is always true if the equations are linear, but unfortunately, almost never true if they are nonlinear. Another approach was the phase plane where one solution is plotted as a function of the other, with time as an implicit variable. Very important characteristics of the solution can often be determined by phase plane methods without actually finding the solution. Finally, the equations can be numerically solved by simulation on a digital computer using Euler's method or some other more efficient algorithm.

B. The Phase Plane

The pair of equations in (7.1) can be reduced to a single equation by eliminating the time variable t. This can be done by simply dividing one by the other to give

$$\frac{dp_1}{dp_2} = \frac{f(p_1, p_2)}{g(p_1, p_2)} \tag{7.9}$$

The solution of this equation is examined in the p_1, p_2 plane, which is called the **phase plane**.

As an example, consider the competition model in (7.7) in the phase plane

$$\frac{dx}{dy} = \frac{a}{c}\frac{x - xy}{y - xy} \tag{7.10}$$

Solutions in the phase plane are

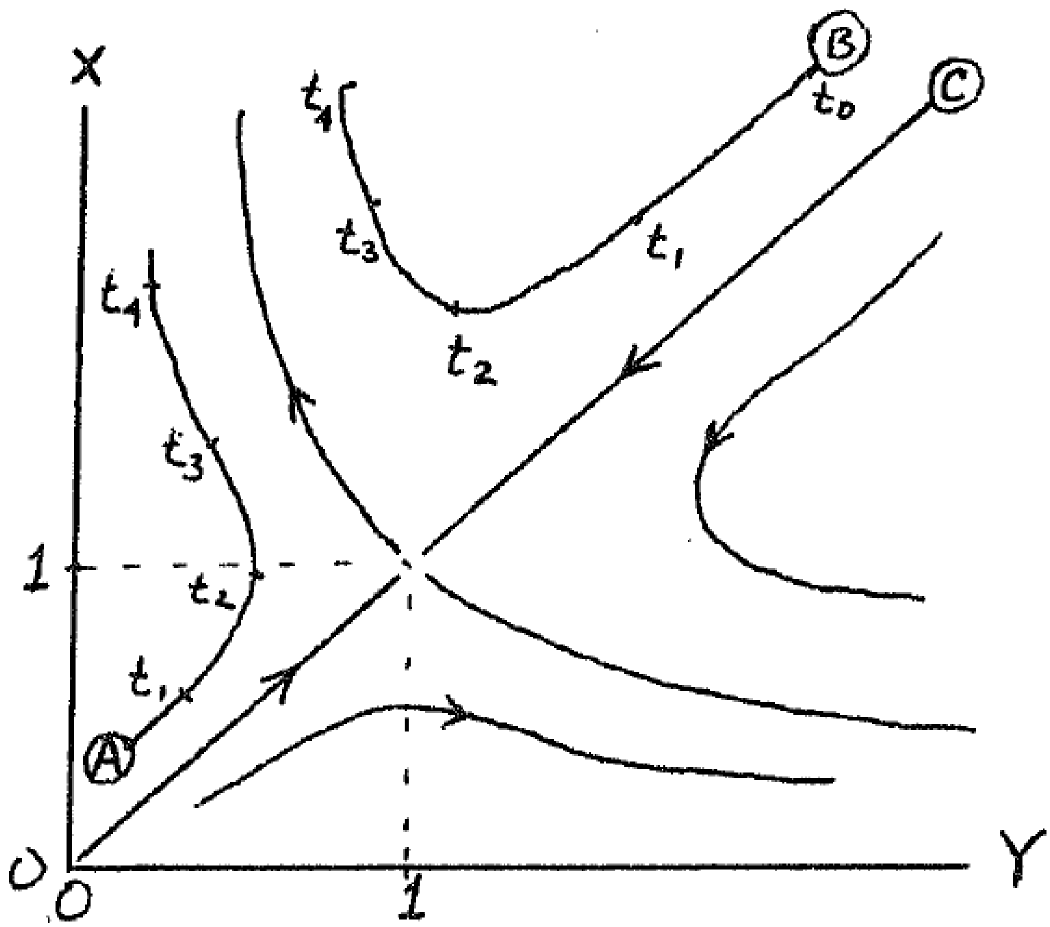


Figure 7.1

The trajectory starting at A gives the value of x and y with time t, an implicit variable, indicated by the values shown. If a different initial mixture of populations had been assumed, e.g., B, then a different trajectory would result. Indeed, any initial mixture is a point on the phase plane, and the trajectories indicate how they evolve in time. The more conventional time solutions are shown for the initial of A by

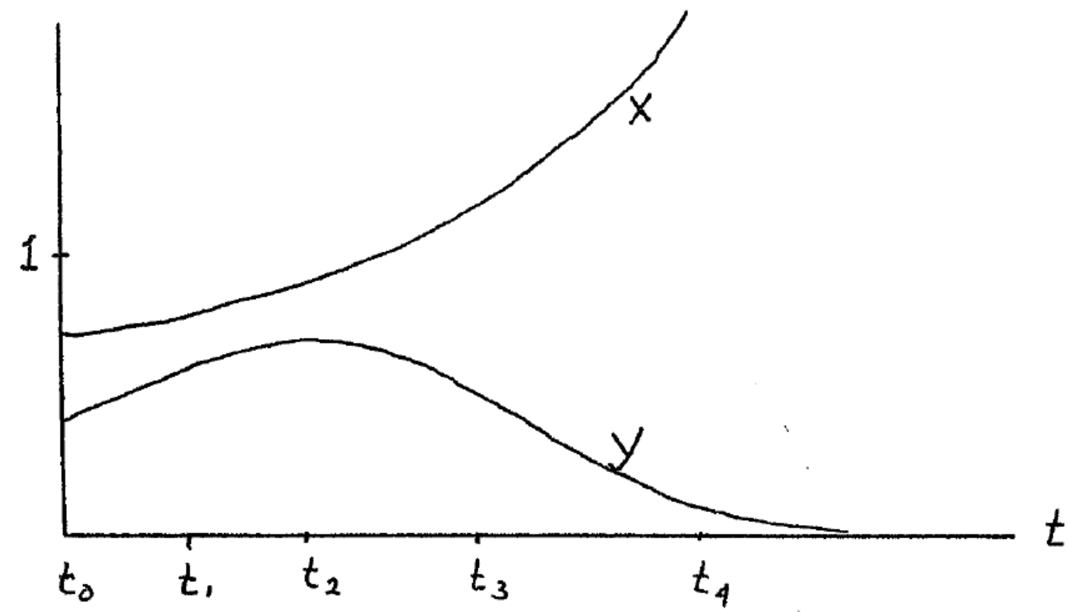


Figure 7.2

and for B by

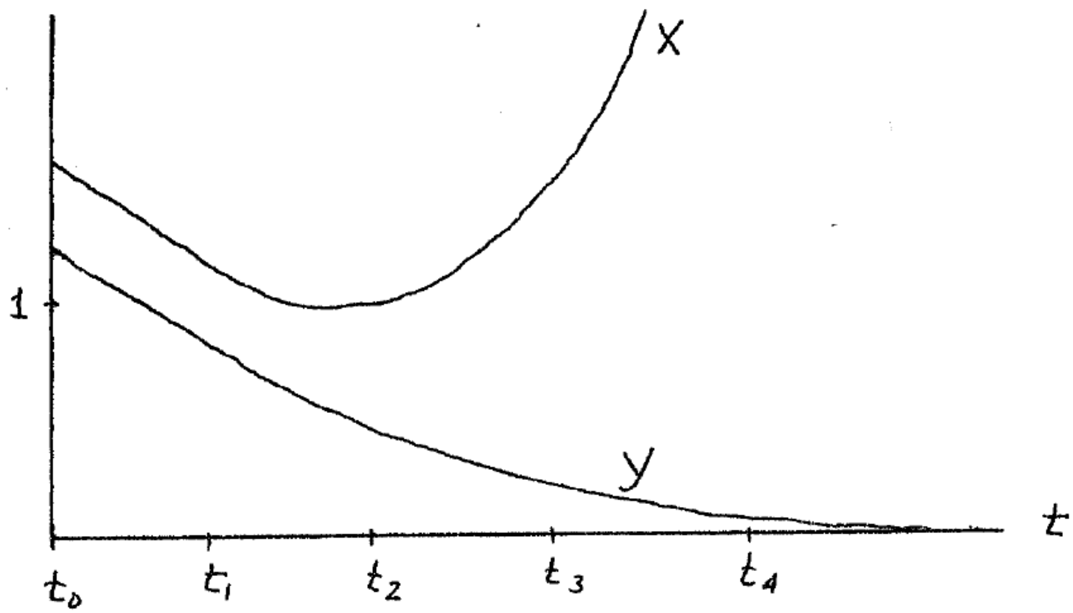


Figure 7.3

Note the relation of the phase plane plots and the time plots. This particular problem will later be examined in greater detail.

One might wonder why this peculiar representation of the solutions is the form of one variable considered as a form of the other. This phase plane approach, although a bit unnatural at first, proves to be a very powerful tool. It is used by many in the literature [19] [22] [24] [2] and is a standard mathematical tool. [4] [6] It is worthwhile developing this concept before analyzing several physical systems.

Note that the phase plane contains all possible time plane plots for various mixtures. It can be shown that if the system has unique solutions, then the phase plane trajectories cannot cross. This means that a few key trajectories can be constructed which will make obvious what all other trajectorie will have to be. For example, in the above competition model, the initial mixture always determines who the eventual winner will be. Any initial mixture to the right of the line from the origin to C results in y increasing without bound and x becoming extinct. Initial mixture to the left gives the opposite result.

There are several procedures that aid in the construction and interpretation of phase plane trajectories. There are special points on the plane known as **equilibrium points** or **singular points** that are important. If both $\frac{dx}{dt}$ and $\frac{dy}{dt}$ in are zero, then x and y are constants and the system is in equilibrium. This means that at these points both the numerator and denominator of (7.9) are zero. For the competition model of (7.10), there is a singular point at $x = 0$ and $y = 0$, and another at $x = 1$ and $y = 1$. Singular points may be stable or unstable depending on whether small perturbations away from the point tend back to it or go away from it. Both points mentioned above are unstable.

A particular informative way of finding the singular or equilibrium points is to consider what are called **partial equilibrium lines** in the phase plane. The curve of all possible solutions of the equition

$$f\ (p_1,\ p_2) = 0 \tag{7.11}$$

is called the **partial equilibrium curve** for population p_1. This is understood by considering the first equation in (7.1) alone. The equation (7.11) implies $\frac{dp_1}{dt} = 0$, therefore, one side of this curve $\frac{dp_1}{dt}$ will be positive and on the other side it will be negative. If a particular $f = 0$ curve was given by

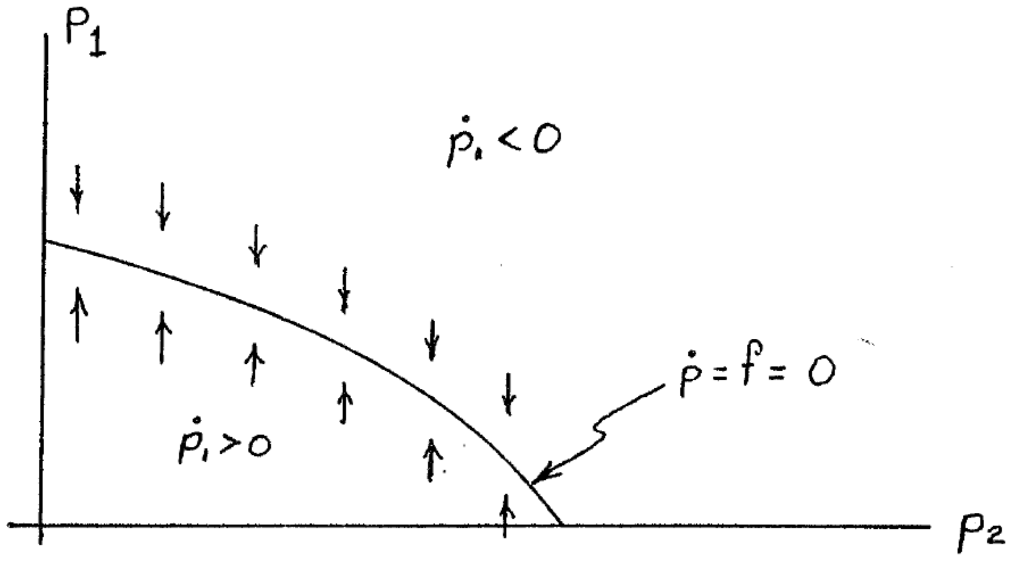


Figure 7.4

for any given fixed p_2, p_1 would move to the $f = 0$ partial equilibrium curve. This curve would, therefore, give the effects of p_2 on the equilibrium values of p_1. In other words, for a system controlled by the first equation of (7.1) if p_2 is given, the $f = 0$ curve will give the equilibrium value $psub1$ approaches. .pp In fact, however, p_2 is not fixed, but must obey the second equation in (7.1). If this equation is examined separately, we have a second curve called the partial equilibrium curve for p_2 given by

$$g\ (p_1,\ p_2) = 0 \tag{7.12}$$

A similar analysis of this equation shows the effects of p_1 on the equilibrium values of p_2, and can be visualized by the following illustration of a $g = 0$ curve.

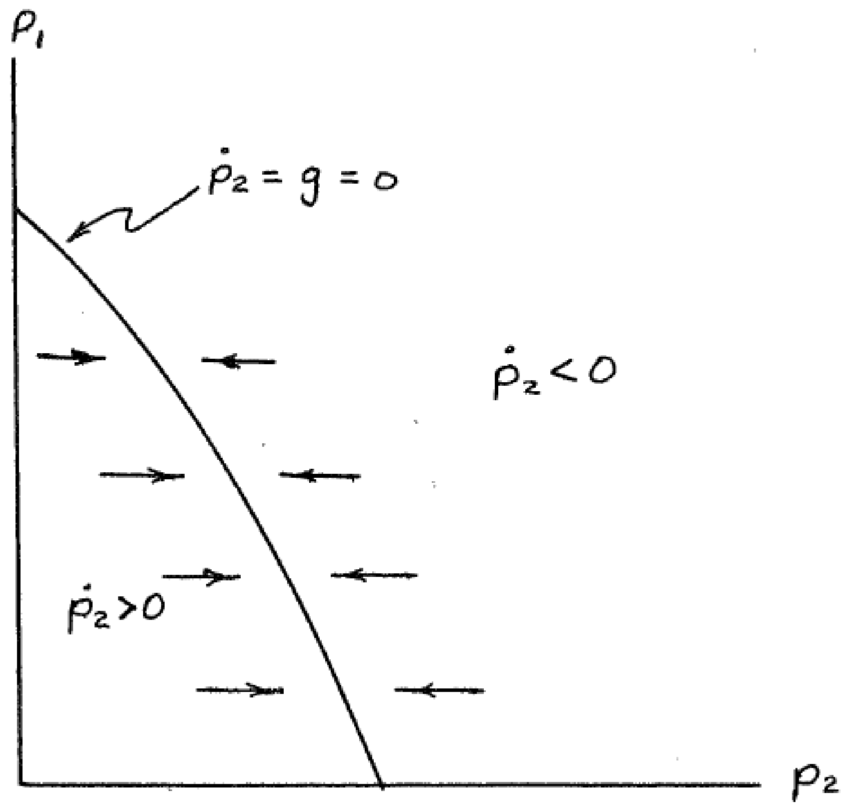


Figure 7.5

If these two curves are considered simultaneously, then not only are the singular points determined by the intersections, but the stability of the points and the nature and direction of the trajectories can be estimited by the signs of $\dot{p}_1$ and $\dot{p}_2$ in the various regions. For these illustrated curves of $f = 0$ and $g = 0$, we have

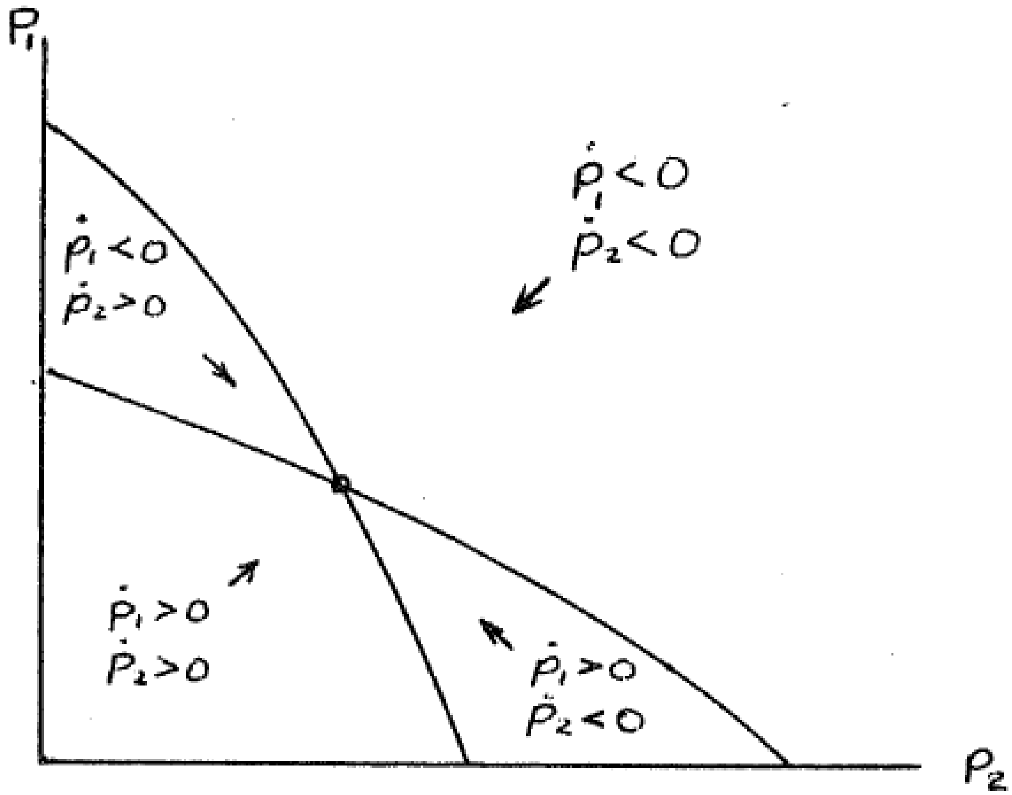


Figure 7.6

This determines the singular point, and the directions show that it is stable. Applying this to the Lotka-Volterra competition model of (7.7) for the partial equilibrium curves gives

$$\dot{x} = a(x - xy) = 0 \tag{7.13}$$

or

$$y = 1 \tag{7.14}$$

and

$$\dot{y} = c(y - xy) = 0 \tag{7.15}$$

or

$$x = 1 \tag{7.16}$$

In the phase plane, these are

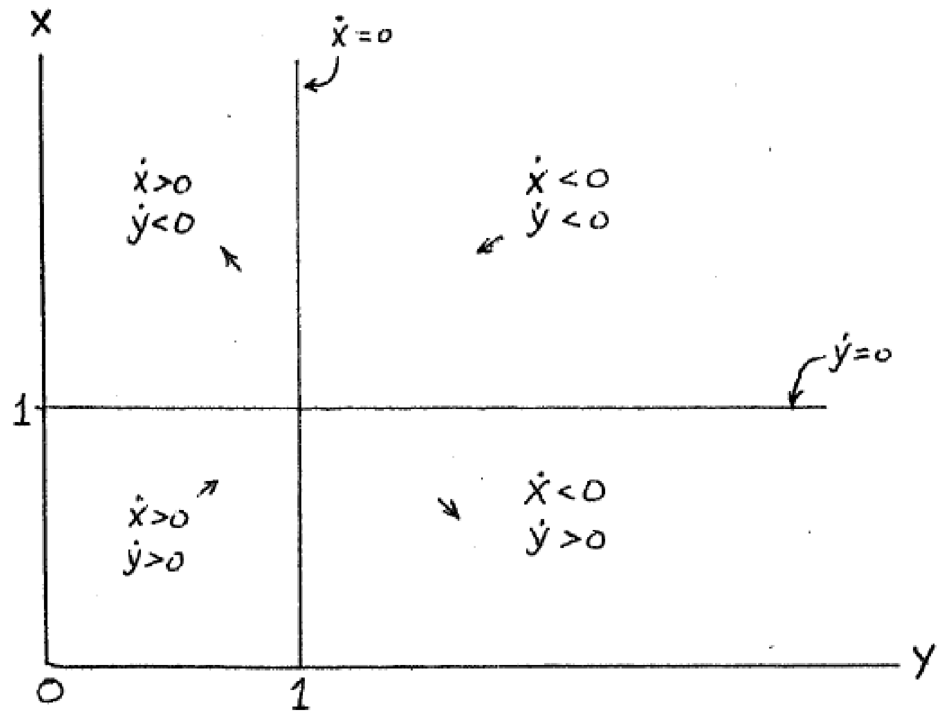


Figure 7.7

Another tool that is very useful and is related to the preceding discussion is the method of isoclines.[4] Here we find curves in the phase plane where all the trajectories that cross that curve have the same slope. The partial equilibrium curves are two isoclines. The $f = 0$ curve implies from (7.9) that the slope of all trajectories along that curve is zero. The slope of all trajectories along the $g = 0$ curve is infinite. If we find the isocline for a slope of m, this is done from (7.9) by setting

$$\frac{f}{g} = m \tag{7.17}$$

For the competition model with $a = c = 1$, we have

$$\frac{x - xy}{y - xy} = m \tag{7.18}$$

Solving for x as a function of y gives

$$x = \frac{my}{1 + (m - 1)\, y} \tag{7.19}$$

The $m = 0$ isocline is $y = 1$. The $m = \infty$ isocline is $x = 1$. The $m = 1$ isocline is

$$x = y \tag{7.20}$$

and $m = -1$ gives

$$x = -\frac{y}{(1 - 2y)} \quad . \tag{7.21}$$

In the phase plane the isocline looks like

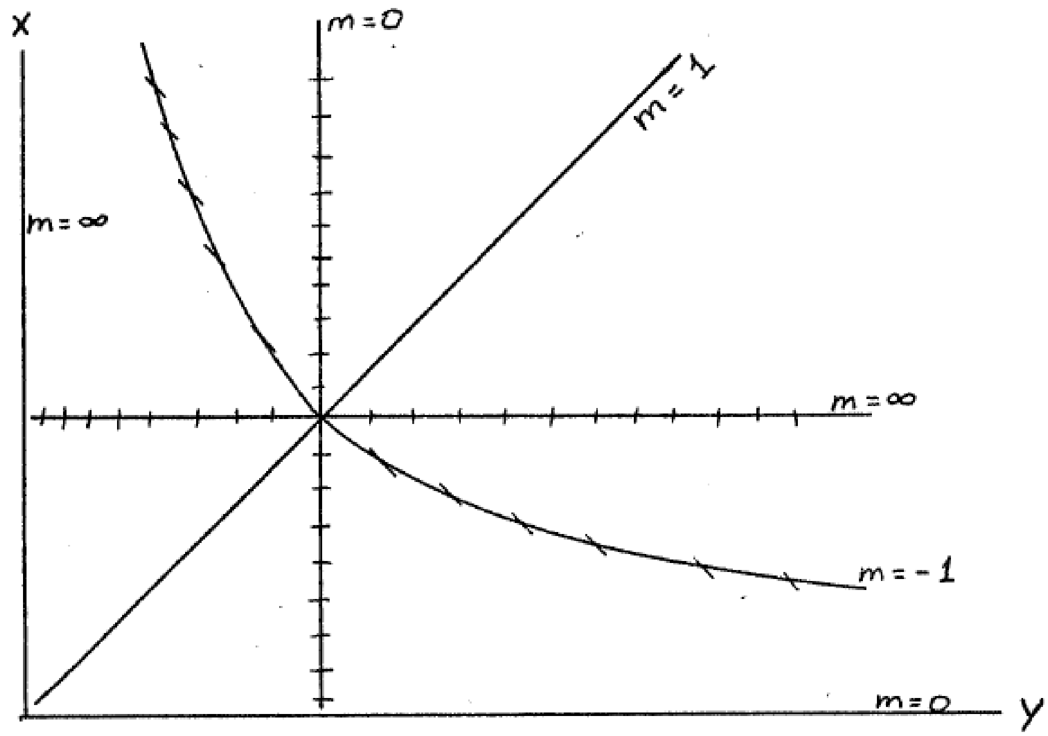


Figure 7.8

Note how the isoclines aid one in sketching or visualizing the phase plane solution trajectories.

This should be enough detail on this approach to allow application to the various two-variable models that can be so interesting.

C. : Competition Models

We will now return to the competition model of (7.3) and examine it in more detail. Consider a situation where the uninhibited growth rate of population p_1 is 10%. This implies $a = 0.1$ in (7.3). Assume that the negative effects of p_2 are such that 100 members of p_2 cancel the positive effects of one member of p_1. >From the first equation of (7.3), we have

$$\dot{p}_1 = ap_1 - bp_1 p_2 = (a - bp_2) p_1 \tag{7.22}$$

If $a = 0.1$, then $b = 0.001$. We also assume that p_2 has the same self-growth rate, and p_1 affects p_2 in the same way that p_2 affects p_1. This gives $c = 0.1$ and $d = 0.001$. The model becomes

$$\dot{p}_1 = 0.1\ p_1 - 0.001\ p_1\ p_2 \tag{7.23}$$

$$\dot{p}_2 = 0.1\ p_2 - 0.001\ p_1\ psb2\ . \tag{7.24}$$

Using Euler's method to convert these differential equations to difference equations, we see that

$$p_1(n + 1) = p_1(n) + T\ a\ p_1(n) - T\ b\ p_1(n)\ p_2(n) \tag{7.25}$$

$$p_2(n + 1) = p_2(n) + Tc\ p_2(n) - T\ d\ p_1(n)\ p_2(n) \tag{7.26}$$

These were programmed on a Tektronix 31 programmable calculator with an automatic plotter to give the phase plane output shown in Figure G. The trajectories were generated by running the simulation with various initial populations. For example, the lowest trajectory was run with an initial population of $p_1 = 25$ and $p_2 = 50$. The next one used $p_1 = 30$ and $p_2 = 35$.

If a different situation is considered where one population has a growth rate of 20% and the other 5%, but the interactions are still at a ratio of 100, the equations become

$$\dot{p}1 = 0.2\ p_1 - 0.002\ p_1\ p_2 \tag{7.27}$$

$$\dot{p}2 = 0.05\ p_2 - 0.0005\ p_1\ p_2 \tag{7.28}$$

The solutions for this case are shown in Figure H. Here the results of the different rates are rather startling. The trajectory number 1 starts at $p_1 = 10$ and $p_2 = 1$, yet p_2 overcomes p_1. Trajectory number 2 starts at $p_1 = 16$ and $p_2 = 1$, and p_2 still wins; but when the initial values are $p_1 = 17$ and $p_2 = 1$, trajectory number 3 shows p_1 wins For $p_2 = 500$ and $p_1 = 200$ or 240, trajectories numbers 4 and 5 show p_2 wins; but with $p_2 = 500$ and $p_1 = 250$ or 300, trajectories 6 and 7 show p_1 wins. This exemplifies the very large difference a four-to-one growth rate ratio can make, and how critical the outcome depends on the initial values. It also illustrate the power of the phase plane in describing the model.

In the basic competition model described by (7.3), and when normalized, described by (7.7), we see that even if the interactive terms are very small, one population always grows without limit and the other becomes extinct. This describes a "survival of the fittest" model, but the unlimited growth and no possibility of coexistence seems unreasonable.

The next level of complication is the addition of a limit to growth in the same manner that the exponential was changed to a logistic. A crowding or self-competition term is added to the simple competition model. Consider now

$$\dot{p}_1 = ap_1 - bp_1\ p_2 - e\ p_1^2 \tag{7.29}$$

$$\dot{p}_2 = cp_2 - dp_1\ p_2 - f\ p_2^2 \tag{7.30}$$

Using the normalizing procedure that was used before on (7.3) reduces the number of parameters from six to four:

$$\dot{x} = ax - axy - kx^2 \tag{7.31}$$

$$\dot{y} = cy - cxy - Ly^2 \tag{7.32}$$

where

$$x = \left(\frac{d}{c}\right) p_1 \qquad k = \frac{ec}{d} \tag{7.33}$$

$$y = \left(\frac{b}{a}\right) p_2 \qquad L = \frac{fa}{b} \tag{7.34}$$

Consider the partial equilibrium curves for this model.

$$f(x, y) = ax - axy - Kx^2 = 0 \tag{7.35}$$

$$x = \frac{(a - ay)}{K} \tag{7.36}$$

$$g(x, y) = cy - cxy - Ly^2 = 0 \tag{7.37}$$

$$x = \frac{(c - Ly)}{c} \tag{7.38}$$

On the phase plane, this becomes

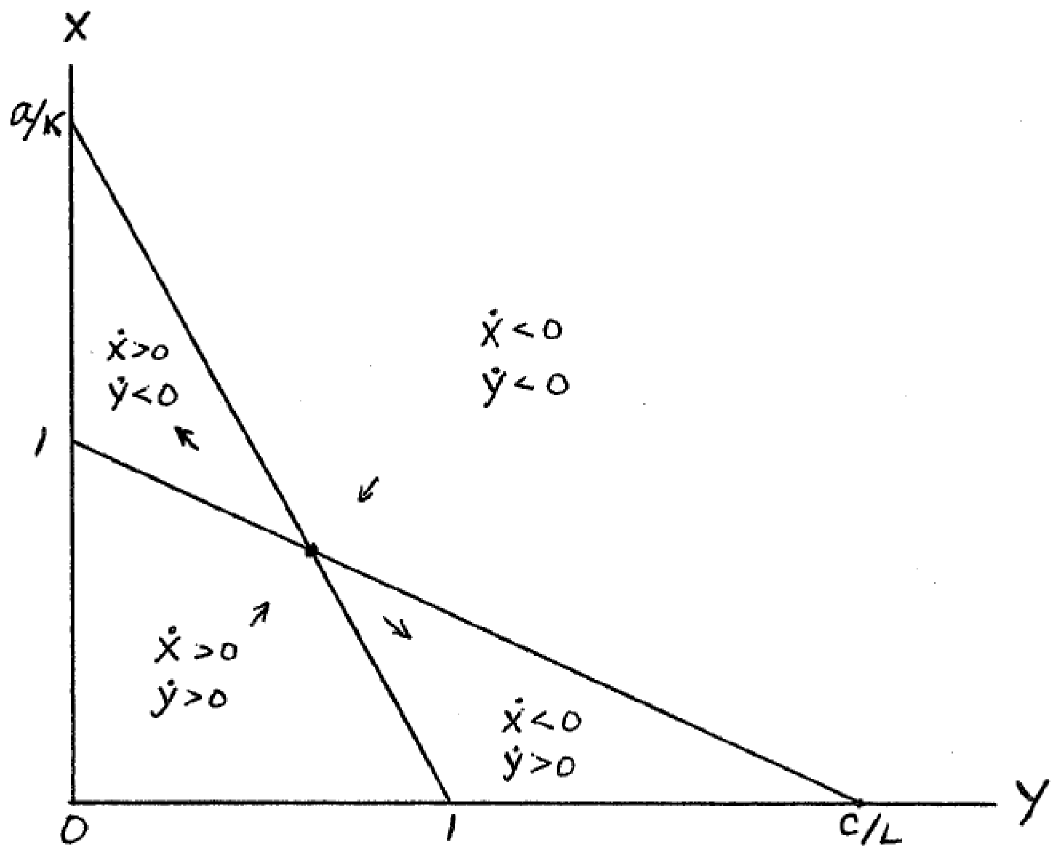


Figure 7.9

It is obvious that the character of this system depends on the relative values of a,b,K and L, and indeed these are from rather different possible systems.

We will first consider the case illustrated above where both limiting factors are relatively small.

$$L < c \qquad \text{and} \qquad K < a \tag{7.39}$$

Note that as K and L approach zero, the system approaches the previously studied system. For this case, there are three possible equilibrium or singular points. There is an unstable point at the intersection of the two partial equilibrium curves, and a stable point at $y = 0$, $x = \frac{a}{K}$, and another stable point at $y = \frac{c}{L}$, $x = 0$. In this case, as before, one or the other population always wins, depending on initial conditions, and the remaining population dies to zero. There is now a limit reached by the winner and indeed, the time plot of the winning population looks very similar to a logistic. For example, for particular initial x and y, we have

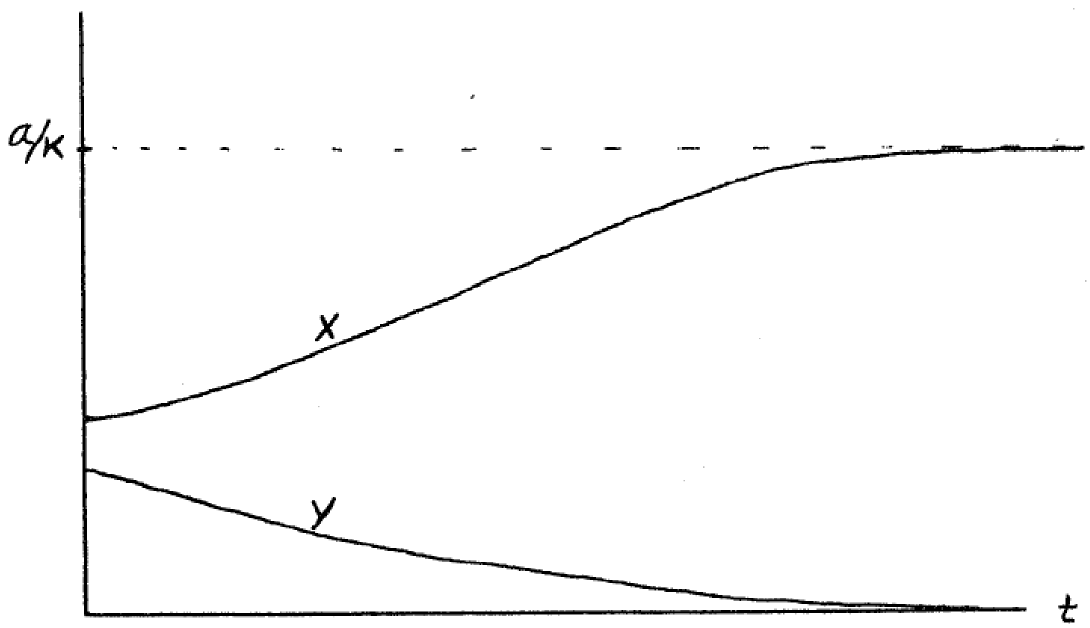


Figure 7.10

The phase plane trajectories are illustrated for the normalized variables x and y in Figure J. The terms a and c are set equal to one, with K and L set equal to one-half. The winning population approaches 2 as its equilibrium value, and the loser becomes extinct.

The second case to consider has strong self-limiting factors relative to the interactive terms

$$L > c \qquad \text{and} \qquad K > a \tag{7.40}$$

The partial equilibrium curves in the phase plane are

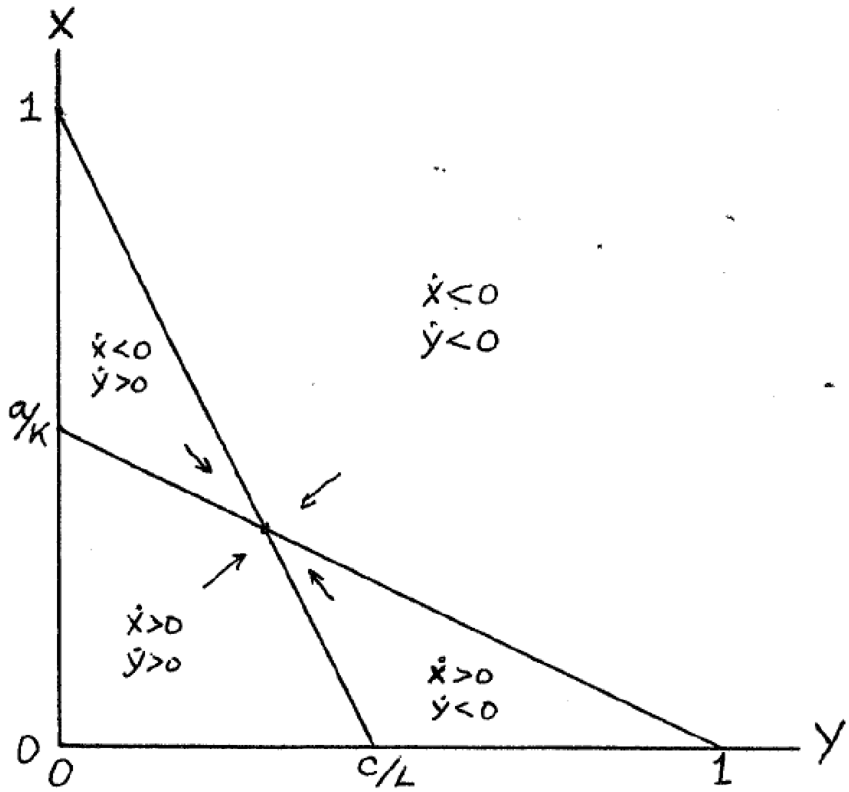


Figure 7.11

Here the signs of the derivatives in the various regions of the phase plane show that there is only one stable equilibrium point at the intersection of the two curves. Here is a case of stable co-existence predicted for a competition model. The phase plane trajectories are shown in Figure K for the normalized variables where $a = c = 1$ and $K = L = 2$.

The third case is not symmetric. It allows one population to have a stronger self-limiting feature, and the other a stronger interactive term. This is given by

$$L < c \qquad \text{and} \qquad K > a \tag{7.41}$$

The partial equilibrium curves in the phase plane are

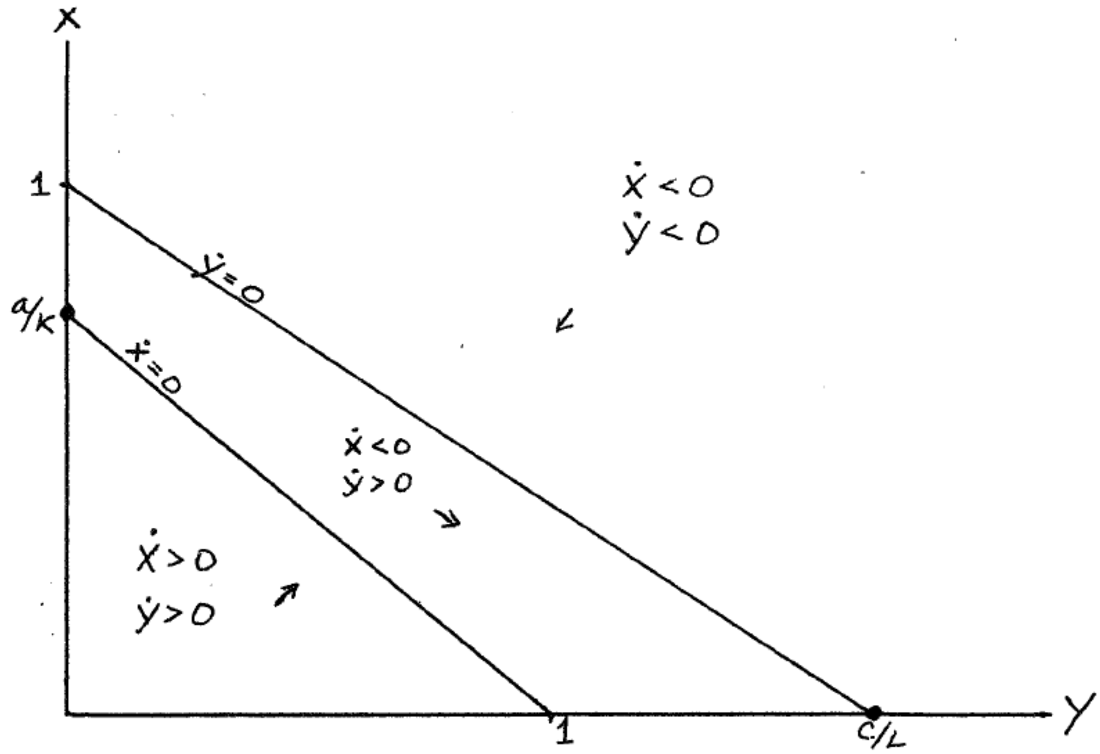


Figure 7.12

The equilibrium point at $x = 0$ and $y = \frac{c}{L}$ is the only stable point. For this case, y always wins for any non-zero initial values, and x always becomes extinct. Figure L illustrates the phase plane trajectories for the case where $a = c = 1$, $K = 2$, and $L = \frac{1}{2}$.

The fourth case is similar to the third, but the roles of the two populations are reversed. The results are similar with x always winning and approaching an equilibrium point of $x = \frac{a}{K}$ and $y = 0$. The phase plane trajectories look like Figure L with the axes reversed.

The use of these competition models can be very interesting in what they say about the effects of the various growth, interactive, and limiting parameters. Applications can be made in short time spans to competing populations in population ecology, or over longer time spans to biological evolution. There are many other possibilities of economic models or international models that could be pursued, but we now turn to a very different type of interaction to be modeled.

D . Predation & Prey Models

If the relation between two populations is not one of competition but one of one population preying on the other, a very different dynamic situation results. We first consider the simple Lotka-Volterra model where (7.1) becomes

$$\frac{dp_1}{dt} = a\, p_1 - b\, p_1\, p_2 \qquad (7.42)$$

$$\frac{dp_2}{dt} = -\, c\, p_2 + d\, p_1\, p_2 \qquad (7.43)$$

This represents a system where p_1 is the population of the prey that has a growth rate a when there are not predators. The parameter b is the negative effect of predation on p_1, and the product $p_1\ p_2$ models the frequency of encounter of the two. The population p_2 is that of the predation who would die out at a rate of c if there were no prey. The coefficient d gives the positive effect of the prey on the predator, and again $p_1\ p_2$ models the frequency of encounter.

As was done before, if the populations are normalized by $x \quad \left(\frac{d}{c}\right) p_1$ and $y = \left(\frac{b}{a}\right) p_2$, then (7.42) becomes

$$\dot{x} = a\,(x - xy) \tag{7.44}$$

$$\dot{y} = -c\,(y - xy) \tag{7.45}$$

The phase plane equation is

$$\frac{dx}{dy} = -\frac{a}{c}\,\frac{(x - xy)}{(y - xy)} \tag{7.46}$$

Using the method of isoclines by finding the curves where the slope is constant shows a remarkable result. First, consider the partial equilibrium curves

$$f\,(x, y) = a\,(x - xy) = 0 \tag{7.47}$$

$$y = 1 \tag{7.48}$$

$$g\,(x, y) = -c\,(y - xy) = 0 \tag{7.49}$$

$$x = 1\ . \tag{7.50}$$

On the phase plane, this is

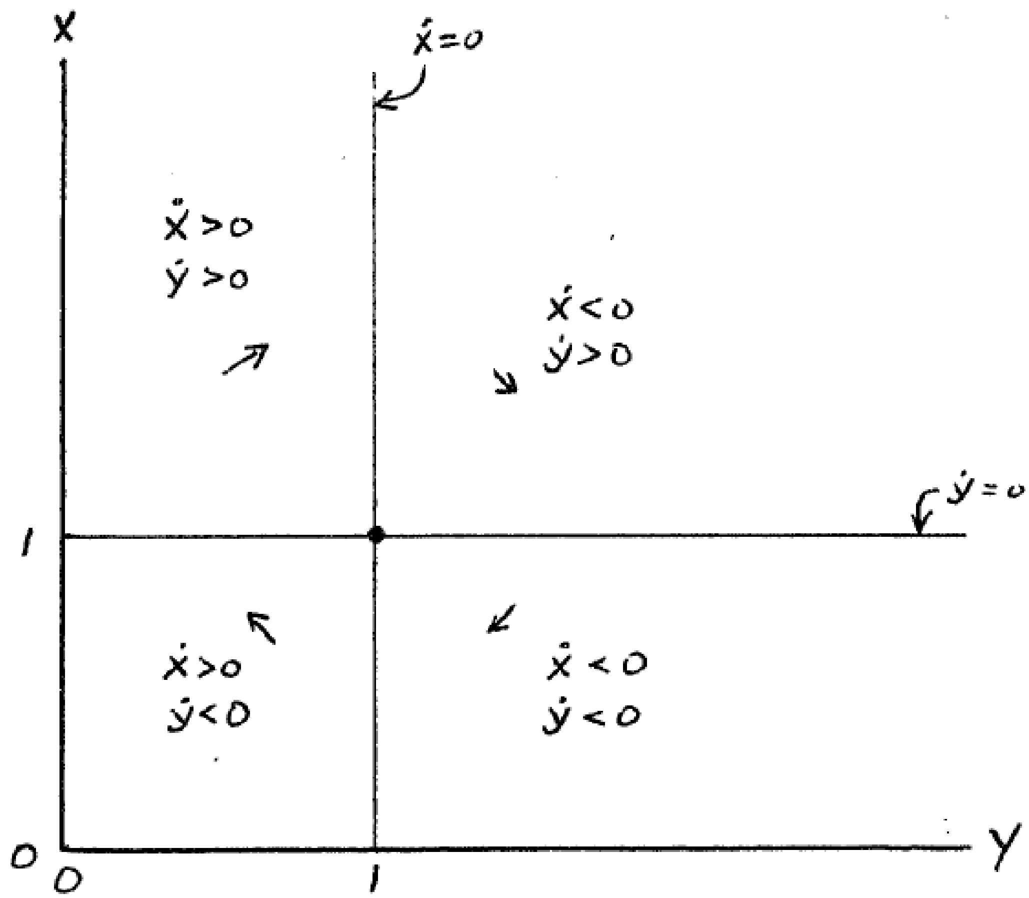


Figure 7.13

The solution trajectories in the phase plane are shown in Figure M. It can be shown [4] that for any positive values of the parameters in (7.42), the solutions in the phase plane are closed nested curves that enclose the singular point. The closed trajectories are called **limit cycles,** and they give rise to periodic or cyclic function when displayed as a function of time. The example used assumed an unlimited growth rate of the prey population p_1 to be 5% per year. The death rate of the predator with no prey is set at 10% per year. The interactive terms are set to be equal to the self rates

when $p_1 = 100$ and $p_2 = 200$. This gives

$$\dot{p}_1 = 0.05\, p_1 - 0.0005\, p_1\, p_2 \tag{7.51}$$

$$\dot{p}_2 = -\, 0.1\, p_2 + 0.0005\, p_1\, p_2 \tag{7.52}$$

There are several interesting features of the solutions. For any initial mixture of population, a limit cycle passes through it. The resulting oscillations have amplitude and frequency that depend on the starting condition, and oscillations neither grow or decay. Unfortunately, the use of Euler's method destroys the exact form of the solutions. Note that the trajectories did not exactly close on the left side of Figure M.

They can be made to approximate the exact solution of (7.51) by choosing T very small – but that slows down the calculations and can sometimes cause other numerical errors.

Time plots of these solutions are shown in Figure N for initial values of $p_1 = 200$ and $p_2 = 50$. In Figure P the initial values are $p_1 = 50$ and $p_2 = 200$. Compare the initial values, maximum and minimum values, with the phase plane trajectories. Note that the period of the oscillation in Figure N is 91 years, and in Figure P, 110 years. The slight increase in amplitude of the oscillations is due to the Euler algorithm, not the model. The time interval T was set at 0.2 years.

There are both interesting theoretical and practical aspects to this model. Serious error can occur when one of the populations is small. Minor variations which are assumed to average out with large numbers, do not. In many experimental verifications of this model, one of the populations will die out at a minimum rather than regenerate. Also, the model is rather sensitive to small errors. The addition of small terms to the basic equation (7.42) causes great change in the character of the solution. This model has been studied in detail by population ecologists. [19] [21] [4] [22] [24]

We will next examine the effects on the simple predator-prey model of adding a crowding term as was done on the competition model and the logistic model. Consider the model

$$\dot{p}_1 = a\,p_1 - b\,p_1\,p_2 - e\,p_1^2 \tag{7.53}$$

$$\dot{p}_2 = -\,c\,p_2 + d\,p_1\,p_2 - f\,p_2^2 \tag{7.54}$$

where the coefficients e and f describe the negative effects of crowding and competition within the population or perhaps cannibalism. These equations can be normalized as done before to the form

$$\dot{x} = ax - axy - Kx^2 \tag{7.55}$$

$$\dot{y} = cx - cxy - Ly^2 \tag{7.56}$$

The effects of the added terms are rather dramatic. The partial equilibrium curves are shown in the following phase plane.

This assumes that $\frac{a}{K}$ 1. The singular points are denoted by a circle. The singular points at the origin and at $x = \frac{a}{K}$, $y = 0$ are unstable, while the one at the intersection of the two curves is stable. The equations were programmed with $a = c = L = 1$ and $K = 0.5$. The trajectories in the phase plane are shown in Figure Q. Compare these results with the derivative signs and singular point locations found above. Note that if $K = L = 0$, the two partial equilibrium curves become vertical and horizontal, giving the same results as found earlier in Figure M. Solutions of this model are shown as a function of time in Figure R for initial values of $x = 1$ and $y = 2$, in Figure S for $x = 0.3$ and $y = 2$, and Figure T for $x = 2.5$ and $y = 0.3$.

Note the relations of these time curves to the phase plane trajectories in Figure Q. In all cases, there are "overshoots" and "undershoots" as the populations interact, but they finally settle down to a constant co-existence that is the same for any initial condition.

The model is changed by removing the limiting term on population y by setting $L = 0$. This causes the y partial equilibrium curve to become horizontal; the resulting phase plane trajectories are shown in Figure U. The results are similar to those in Figure Q, but there is more oscillation before the final equilibrium is reached. If a limiting factor is made large by setting $L = 4$, the phase plane trajectories of Figure V result, giving very little oscillation.

A rather different situation results if the parameters are such that $\frac{a}{k}$ 1. In this case, the intersection of the partial equilibrium is in the second quadrant which has no physical meaning. In the first quadrant where populations are positive, the equilibrium point at $x = \frac{a}{K}$, $y = 0$ is the only stable one. This was programmed for $ac = 1$ and $K = 2$. The phase plane trajectories are shown in Figure W, and the time solution for initial values of $x = 1.5$ and $y = 1$ in Figure X.

For these conditions, the predator dies out and the prey self-limits in a manner similar to the logistic.

By choosing more complicated interaction functions for the $f(p_1, p_2)$ and $g(p_1, p_2)$, it is possible to obtain other types of solutions. For the simple case with no limiting in (7.42), the partial equilibrium curves were vertical and horizontal straight lines and the trajectories were closed. With limiting added in (7.53), the curves remained straight, but were no longer ecessarily vertical or horizontal, and the solution trajectories were no longer closed, but would either spiral or smoothly move to an equilibrium point. Although not illustrated here, it is possible to use a model of limiting similar to that will cause a single stable limit cycle to occur, that all trajectories starting outside of it would spiral in to it, and all starting inside of it would spiral out to it. This would give a steady-state oscillation as a time function. Perhaps this type of model could be used to explain some of the cyclic variations that occur in business and economics. Much more work could be done on both the mathematics and interpretation of this predator-prey type system, but we will move on to others now.

E. Simple Non-renewable Resource Model

Here we assume a simple system consisting of a population y that depends on, and consumes, a resource that cannot be replaced. The equations are somewhat similar to the predator-prey model, but the prey could grow and the resource here cannot. If x is the amount of resource, and it is distributed in such a way that the consumption by y is modeled by the product xy, the normalized equations are:

$$\dot{x} = axy \tag{7.57}$$

$$\dot{y} = -cy + cxy \tag{7.58}$$

The partial equilibrium curves in the phase plane are

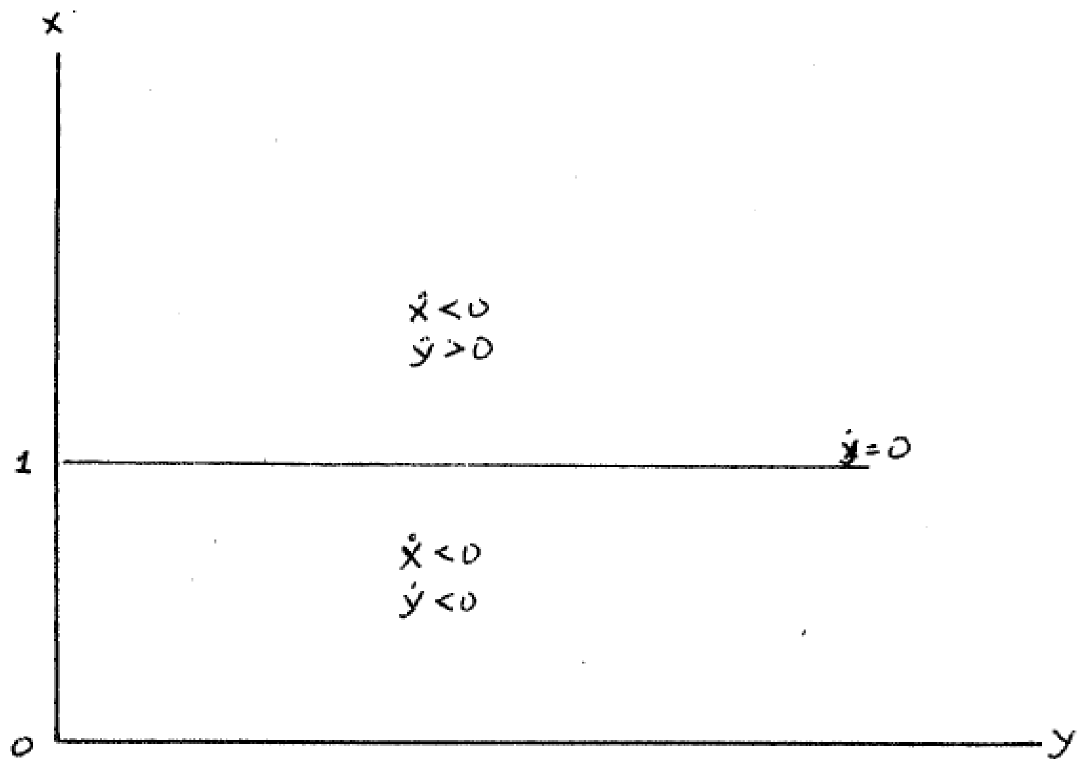


Figure 7.14

The phase plane trajectories are shown in Figure Y. This uses $a = c = 1$.

Note that the resource monotonically decreases while the population may or may not initially increase, but in any case, it ultimately dies out. An interesting result of this model is that there is some resource left after the population is gone. This is caused by the assumption that the distribution is such that consumption is governed by the product xy. If it is assumed that the resource is easily accessible, a better resource model might be

$$\dot{x} = -ay \tag{7.59}$$

The nature of the solutions of this system is left to the reader.

F . An Arms Race Model

We will now move into rather different systems to see how models might be applied. The history of application of dynamic models to problems such as national armament is fairly new, but perhaps older than most realize. [3][25]

A simple linear model is the following

$$\dot{x} = -ax + by + f \tag{7.60}$$

$$\dot{y} = -cy + dx + g \tag{7.61}$$

where the state variables x and y are measures of the arms level of two nations. The coefficients a and c are measures of confidence or expense that cause a decrease in military expenditures; b and d are the effects of the opponent's arms level on ones military build-up. The constants f and g represent the minimum level that would be maintained even if the opposition disarmed.

There are two general cases possible. If the situation is such that $ac>bd$, then the partial equilibrium curves look like

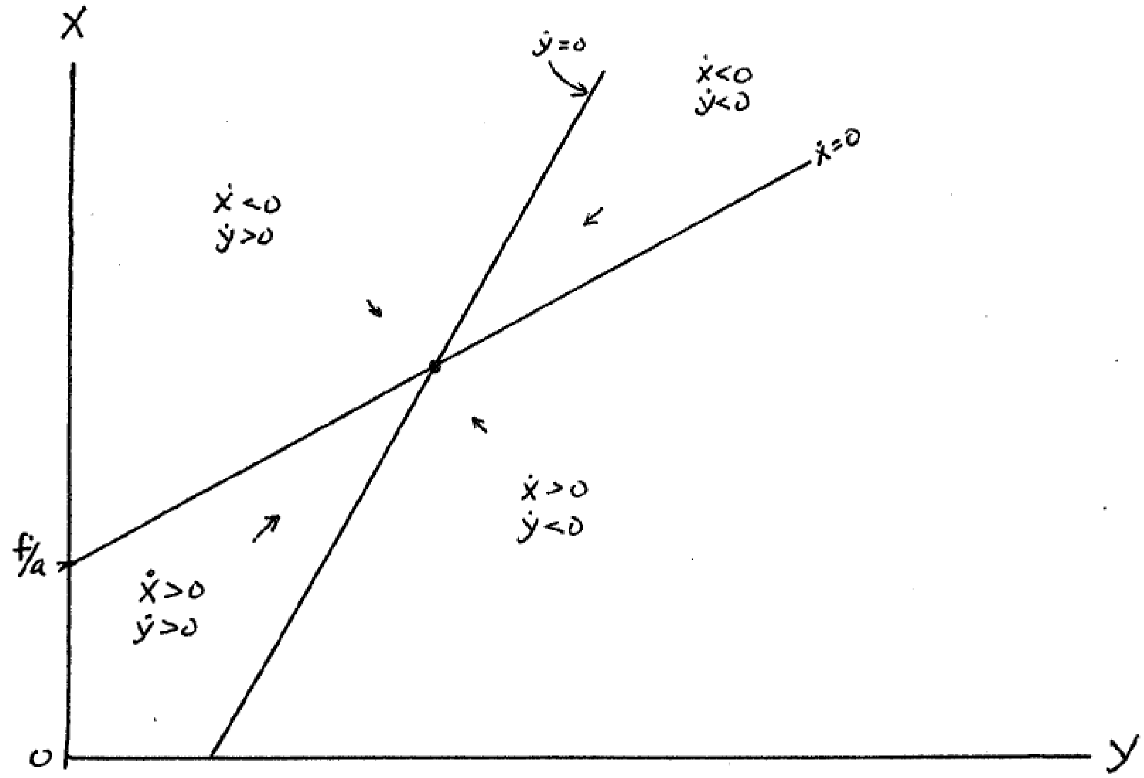


Figure 7.15

The signs of the derivative show that the singular point is stable. This states that the arms race stops at a stable level for this case. If, on the other hand, the conditions are such that $bd>ac$, the curves look like.

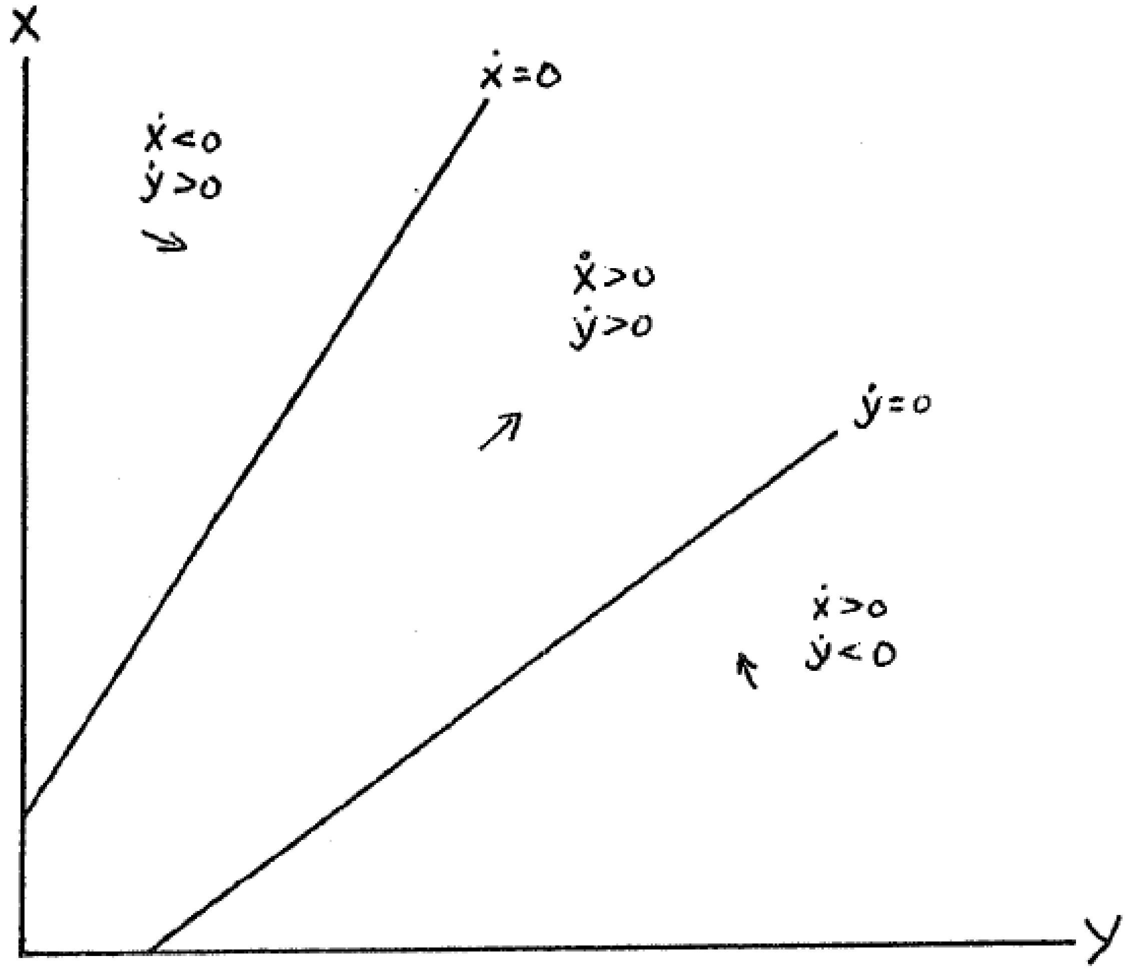


Figure 7.16

Here, there is no stable point, and the armament levels of both nations increase without limit.

This model is linear so that analytical expressions for the solution can be found, and computer simulation is unnecessary. On the other hand, the model is too simple to be realistic, and a more reasonable one would be nonlinear. Again, while these models can be interesting, they leave out too many other state variables to be used for more than gaining insights.

G. Models of Hostility and Friendliness

While it is certainly easier to model and measure quantitative variables such as population, food, or money, it is also possible to apply dynamic modeling techniques to more subjective variables involved with attitudes and feelings. These variables must be quantified in some way that is obviously going to be somewhat subjective. Even though this process is difficult and subject to challenge, it must be done if more complete models of social systems are to be developed. This becomes apparent when, in trying to chose the state variables for a system, it is necessary to know how a group of people feel about another variable to predict their actions. One accepted example is the practice of assigning a monetary value to the **good will** of a company.

As an example, we will consider the dynamics of feelings between two populations in terms of their friendliness or hostility. [2] Let x be the measure of friendliness of population p_1 toward population p_2, and y the friendliness of p_2 toward p_1. Negative friendliness is considered hostility. The equations are

$$\dot{x} = f(x, y) \tag{7.62}$$

$$\dot{y} = g(x, y) \quad . \tag{7.63}$$

The determination and interpretation of f and g is a bit more difficult here. Recall from Section B the definition of partial equilibrium curves. The $f(x,y) = 0$ curve in the phase plane gives the equilibrium values of x for a fixed y. For this model, $f(x,y) = 0$ gives the degree of friendliness x that will be approached by p_1 for an independently set amount of friendliness y of p_2 toward p_1. Consider the following case:

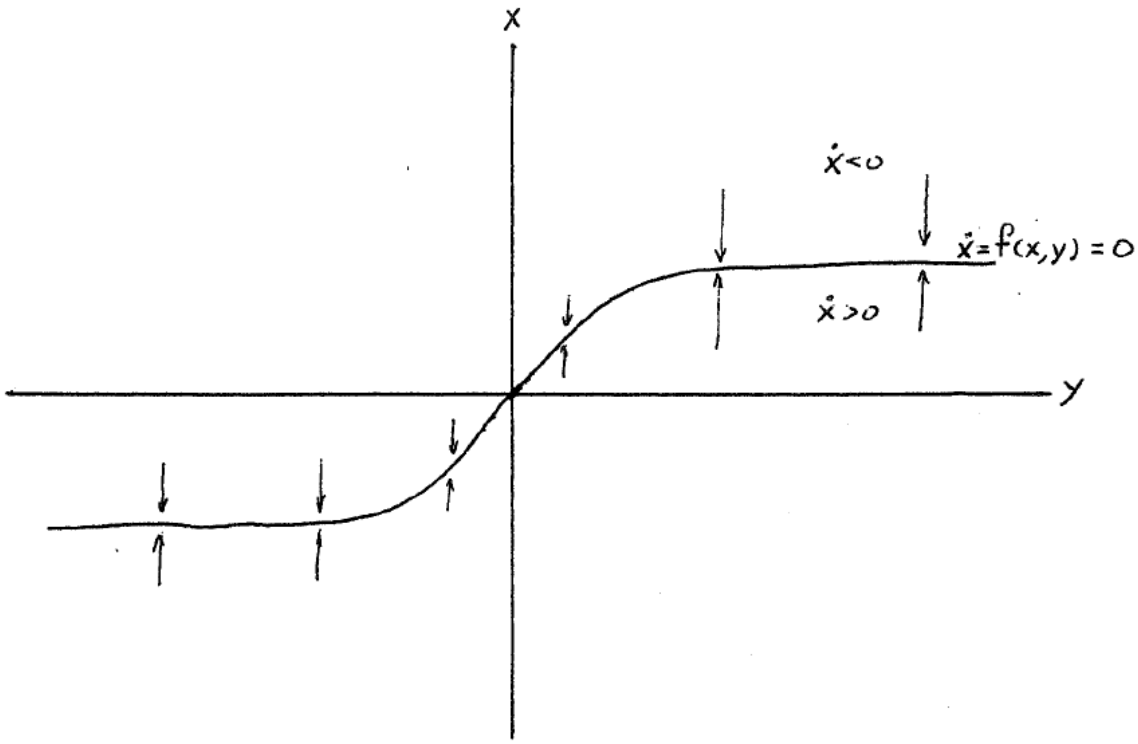


Figure 7.17

Here, if y is neutral, then x become neutral. If y is friendly, then x is friendly. As y becomes more friendly, x increases to a point and then levels off at a maximum amount of friendliness. As y becomes hostile, x responds likewise and finally levels off at a maximum amount of hostility.

Now consider the $g(x,y) = 0$ curve which is the p_2 response to p_1. This is shown by

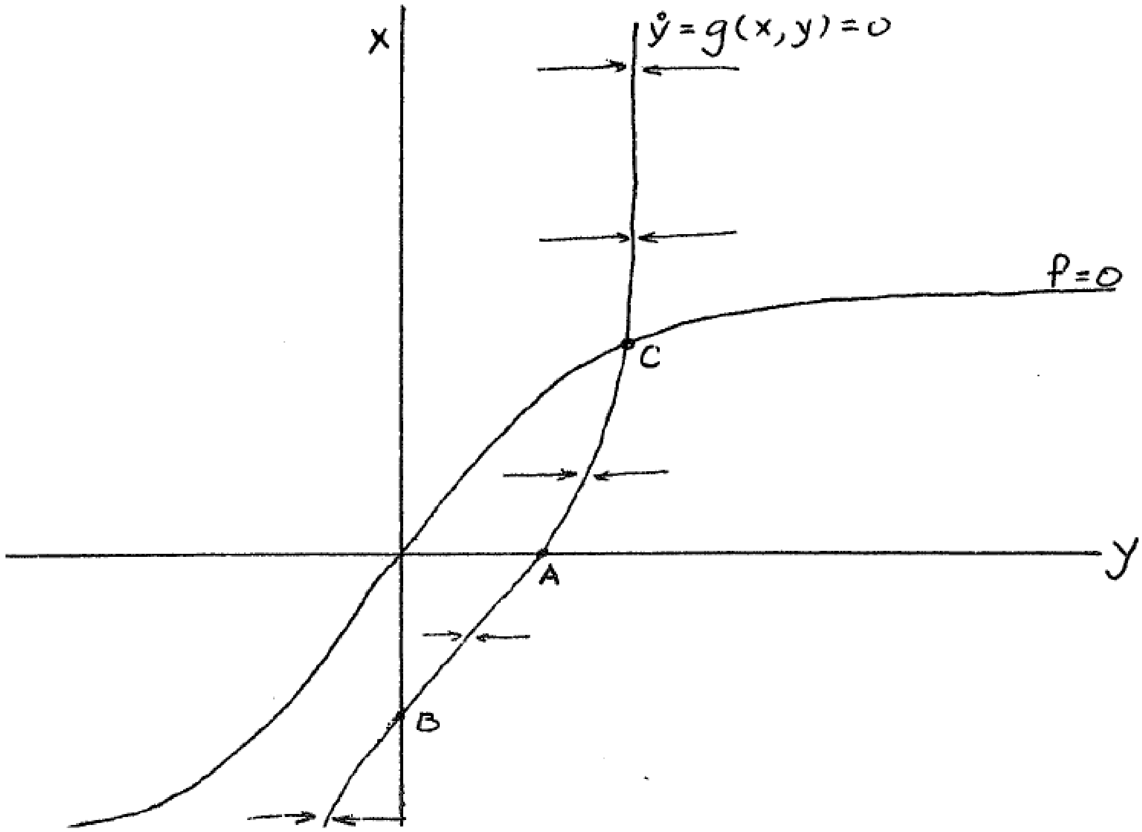


Figure 7.18

Population p_2 is naturally more friendly than p_1. It is friendly even if p_1 is neutral, as is shown by point A. Only after p_1 becomes fairly hostile does p_2 begin to return with hostility as shown by point B.

Given these relations and considering the signs of the derivatives, it is seen that the singular point at C is a stable equilibrium point.

Consider a different set of characteristics where a population would initially return a large amount of hostility a hostile opponent, but after a certain level, would submit or surrender by having a reactionary response to a very hostile situation. This might be described by

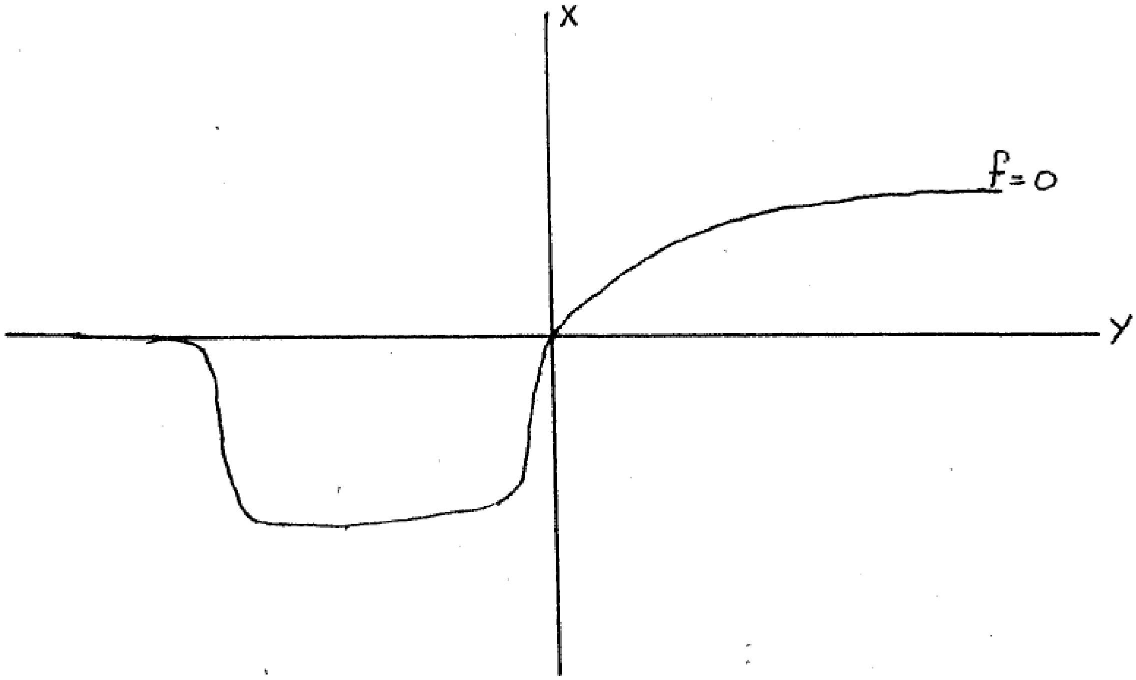


Figure 7.19

Another characteristic is to have almost no response up to a certain level, then to react suddenly as described by:

Many interesting models can be posed and the resulting solutions examined in the phase plane. In some cases, the results are insensitive in the sense that small changes in the partial equilibrium cause only little change in the solutions. Other cases are very sensitive.

A very important aspect of this approach to modeling is the dynamic description. When the trajectories or time solutions are found, not only are the equilibrium points found, but how they are reached is predicted. In some cases, the effects of a .... more important than the final value. Also, sometimes the time necessary to achieve a certain condition is as important as the condition itself.

H. Malthus Revisited

Reading, in this day and time, the 1798 essay by T.R. Malthus, one is struck by both the insight and the naite of this very influential statement. [25] Malthus saw the possible dire consequences of an increasing population in a finite environment. His predictions of doom were based on the assumption that the world population increases according to a geometric sequence, while the food increases according to an arithmetic sequence. Stated in our terms, he assumed population increases exponentially and food linearly. The fact that world food production has more or less kept up with the population has lead critics to discount Malthus and his followers as irrelevant **doomsday prophets.** In fact, some feel this pessimistic view is not irrelevant, it is dangerous. The strong critics, known as **technological optimists,** assume that the factors that have prevented Malthus' predictions from coming true so far will continue to so do. [14] In fact, they claim that growth is not the problem, it is the solution. Growth has given us the highest material standard of living by our abilities of growing faster than our problems.

Our purpose in this treatise is not to take sides, but to suggest a different way of looking at our situation that will give more understanding and insight. Malthus based his prediction on observed and assumed growth patterns of population and food. If one looks at the underlying models that might support these assumptions, the population might come to be modeled by an equation similar to (8). One is hard-pressed to explain his assumed food growth, and that has indeed been the flaw in his assumptions.

The **technological optimists** likewise have implied models to obtain their predictions. The costs of continued economic and technical growth must be considered in any realistic model. A bit of reflection shows that the question is not whether to use a model or not, but to determine what kind of model to use and whether it will be examined and debated explicitly.

Very interesting results have been obtained in the preceding sections on applying two-variable models to various systems of interest to us. It is a valuable exercise to try to better model world population and food than Malthus did. Very soon one discovers (or should discover) that the systems are too complicated to be described by any two-state variables. Unfortunately, when using higher order models, many of our analytical methods no longer work. We lose the powerful tool of the phase plan. Second-order systems are useful to gain insight into simple systems with relatively simple interactions, but now we will have to rely primarily on computer simulation to give solutions of the resulting third and higher order models.

Chapter 8

Higher Order Model[1]

8.1 Higher-Order Models

Once it becomes necessary to include more than two state variables in a model, and if the interactions are nonlinear, the analytical and phase plane techniques can no longer be used. This section will consider several higher-order models and use digital simulation as the tool for analysis. The first example will be a rather logical extension of some of our earlier population models.

A. Population Models with Age Specific Birth and Death Rates

Even cursory examination of the assumptions behind the population model assumed in Section IV show them to be unrealistic. The model

$$\frac{dp}{dt} = rp \qquad (8.1)$$

assumes r to be the difference between the birth rate and death rate, and that these rates are not a function of time or population.

An improvement on this model would allow different birth and death rates to be assigned to members of the population of different ages. This means that the population will have to be divided into groups with similar rates, and that the number of groups necessary will be the number of state variables required.

For example, let p_1 be the population of people between zero and ten years of age, p_2 the population of those from eleven to twenty, p_3 those from twenty-one to thirty, etc. Let b_1 be the average birth rate of p_1, and b_2 the rate for p_2, etc., with d_1 being the average death rate of p_1, etc. Assume the maximum possible age to be one hundred. The equations for this model are given by

$$p_1(n+1) = b_1 p_1(n) + b_2 p_2(n) + \dots b_{10} p_{10}(n) \qquad (8.2)$$

$$p(n+1) = (1-d_1)\, p_1(n)$$

$$p_2(n+1) = (1-d_2)\, p_2(n)$$

$$\cdots p_{10}(n+1) =$$

where the time interval represented by each successive value of n is the same as that for the age span for each population group, i.e., ten years. Likewise, the birth and death rate are numbers per ten-year period. The equations in (8.2) can be described by a flow graph, illustrated below for only three sections.

[1] This content is available online at <http://cnx.org/content/m18166/1.2/>.

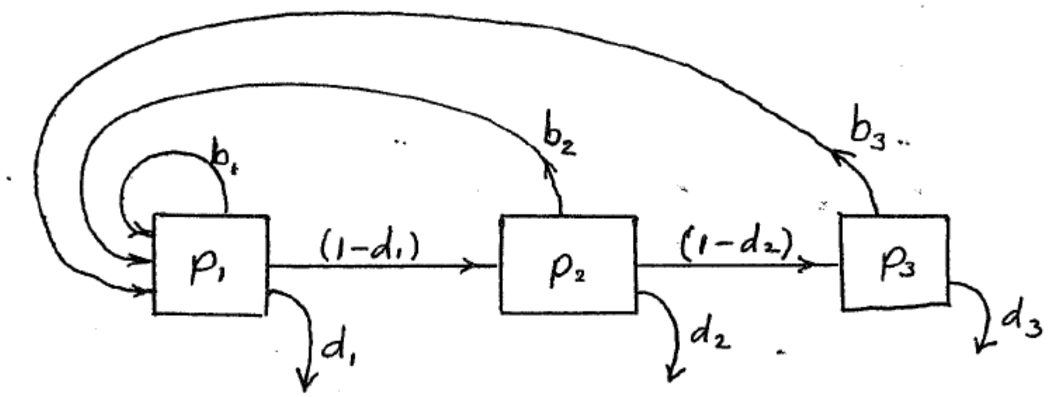


Figure 8.1

These equations can be easily programmed and solved on a computer, but because they are linear, there are some interesting properties that can be worked out analytically. They are best seen by writing as a matrix equation.

$$[\ p_1(n + 1)\, p_2(n + 1)\, p_3(n + 1)\, p_4(n + 1)\, 3dotp_{10}(n + 1)\ ] = \quad (8.3)$$
$$[\ b_1(1 - d_1)\ \ 003dot0b_20(1 - d_2)\ \ 0\ 0......\ \ ...b_90\ \ above\ \ (1 - d_9)\ \ b_{10}0\ 3dot\ 0\ ] \quad [\ p_1(n)\, p_2$$

In compact vector notation, this becomes

$$\underset{\_\ [U+0332]}{P}(n + 1) = A \underset{[U+0332]\ \_}{P}(n) \quad (8.4)$$

>From this expression, it is easily seen that the population distribution after n times ten years from some initial population distribution $Punder$ (0) is given by

$$\underset{\_\ [U+0332]}{P}(n) = A^n \underset{[U+0332]\ \_}{P}(0) \quad (8.5)$$

There are several interesting observations for the readers with a knowledge of matrix theory. After several steps of n, the age distribution will assume a form given by the eigenvector of the largest eigenvalue of A.

After several steps, the age distribution will stop changing and this eigenvector is called the **stable age distribution,** and this largest eigenvalue is the stable growth rate if the eigenvalue is greater than one (decay rate if is is less than one). The problem is a bit more complicated if the eigenvalues are complex (where oscillations occur).

It is possible to modify the equations of (8.2) to allow for shorter T in Euler's method than the span of ages in a population group, and to allow different spans for different groups. Let T be the time interval between each calculation of new levels. Let S_i be the span of ages for the group p_i. Consider the population group p_2. During one time interval T, the number that dies is $d_2\ p_2$, the fraction that advances to the next group is $\left(\frac{T}{S_2}\right)$ times those that do not die $(1 - d_2)\ p_2$, and the rest $\left(1 - \frac{T}{S_2}\right)(1 - d_2)\ p_2$

stay in the group. The equations become

$$P_1 = (n+1) = b_1 p_1 (n) + b_2 p_2 (n) + ... \tag{8.6}$$

$$P_2 (n+1) = \left(\frac{T}{S_1}\right) (1 - d_1) P_1 (n) + (1 - d_2) P_2 (n) \tag{8.7}$$

$$P_M (n+1) = \left(\frac{T}{S_{M-1}}\right) (1 - d_{M-1}) P_{M-1} (n) + \left(\frac{1-T}{S_M}\right) (1 - d_M) P_M (n) \tag{8.8}$$

The birth rates b_i are the number of births per initial value of population in a group p_i per time interval T. The death rate is likewise per time interval T.

This is a rather general formulation that allows non-equal age grouping and short time interval without requiring a high order. The system can be posed in matrix form as before. The main limitation on this approach is that it is linear. In general, the various birth and death rates will depend on crowding and other environmental and social factors that are assumed constant here. Even so, insight can be gained into population growth by experiments on these simple linear models.

B. A Model of the World

One of the most interesting and controversial applications of dynamic modeling is the work of J. Forrester at Massachusetts Institute of Technology on a simulation of the world. In 1970 at the request of an international group called the Club of Rome, Forrester developed a fifth-order model of the work using what he calls "system dynamics," methods that had previously only been applied to industrial and urban systems. The preliminary results were published [13] in 1971, and further work done by his colleague Dennis Meadows was published [10] in 1972. The response to this work was incredible. There has been a flood of articles in newspapers, popular magazines, and scholarly journals – some in praise and others in condemnation. Most have been superficial and emotional. There is, however, one interesting serious response published by a group in England [15] in 1973.

The state variables chosen by Forrester are:

N	population
C	capital
A	agriculture
P	pollution
R	non-renewable resources

Table 8.1

The model then assumed the form

$$\begin{aligned} \dot{N} &= f_1 (N, C, A, P, R) \\ \dot{C} &= f_2 (N, C, A, P, R) \\ \dot{A} &= f_3 (N, C, A, P, R) \\ \dot{P} &= f_4 (N, C, A, P, R) \\ \dot{R} &= f_5 (N, C, A, P, R) \end{aligned} \tag{8.9}$$

In one sense, this work is a logical extension of the dynamic modeling discussed in the earlier sections of this paper, and Forrester's formalism is nothing more than using Euler's method to solve simultaneous differential equations. In another sense, his bold use of these methods represents a distinct departure from the specialized models that the demographer, economist, etc. have used in their separate disciplines.

There are several features of Forrester's approach that should be understood. The functions f_1 , f_2 , etc. are developed in a complicated way using the theories and empirical results of specialists in those areas. This generally means that there are numerous intermediate variables defined and used, both for insight and because the data occurs that way. One should not confuse these with the state variables, however. Forrester also uses tables rather than functions to implement the f_i in the simulation. These are usually easier to handle by non-mathematicians, and again in the form that empirical relations are often known.

A version of the world model has been programmed in APL on an IBM 370 at Rice. The details of this program and instructions on its use are included in the appendix. Examples of the results of the model are given in [13] and [10] and a criticism in [15].

Chapter 9

Supplementary Figures[1]

Figures for the module on second order systems.

[1] This content is available online at <http://cnx.org/content/m18167/1.2/>.

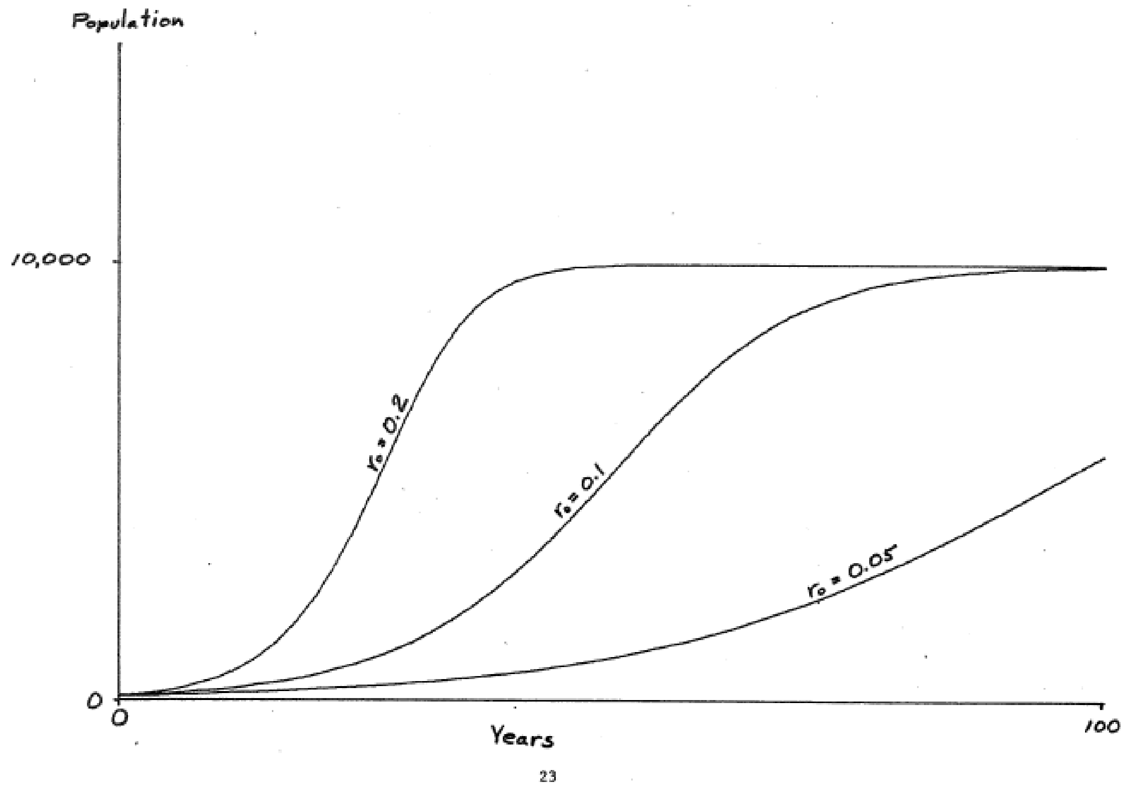


Figure 9.1

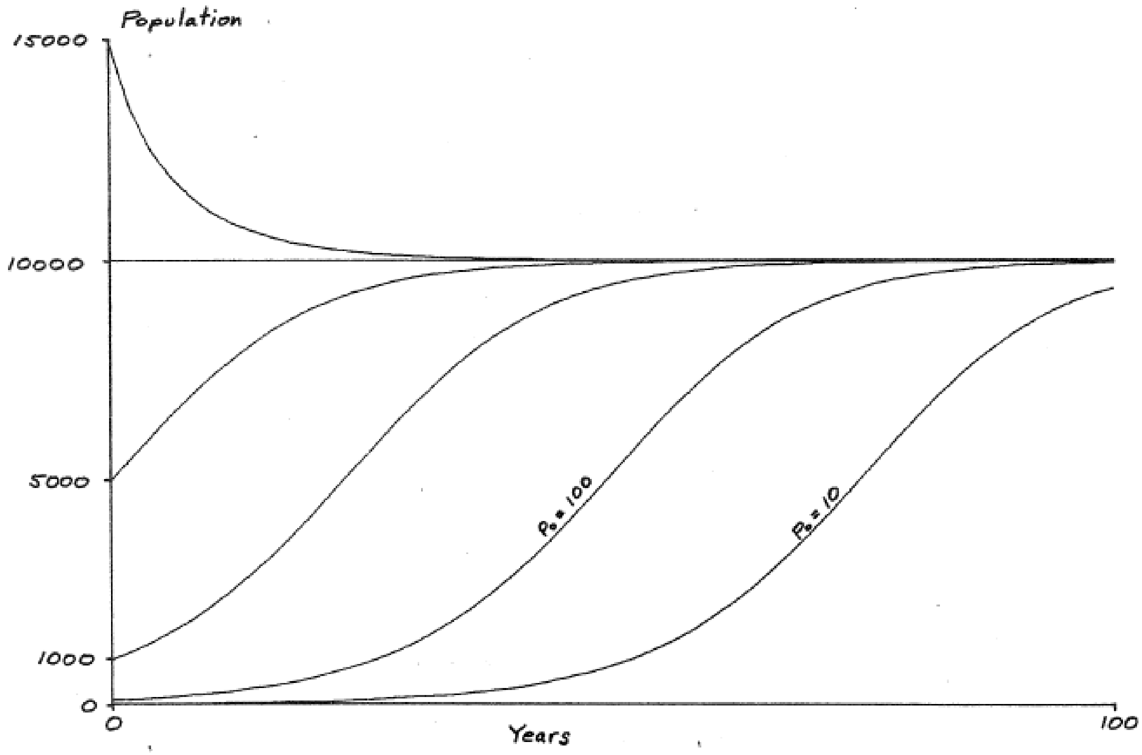


Figure 9.2

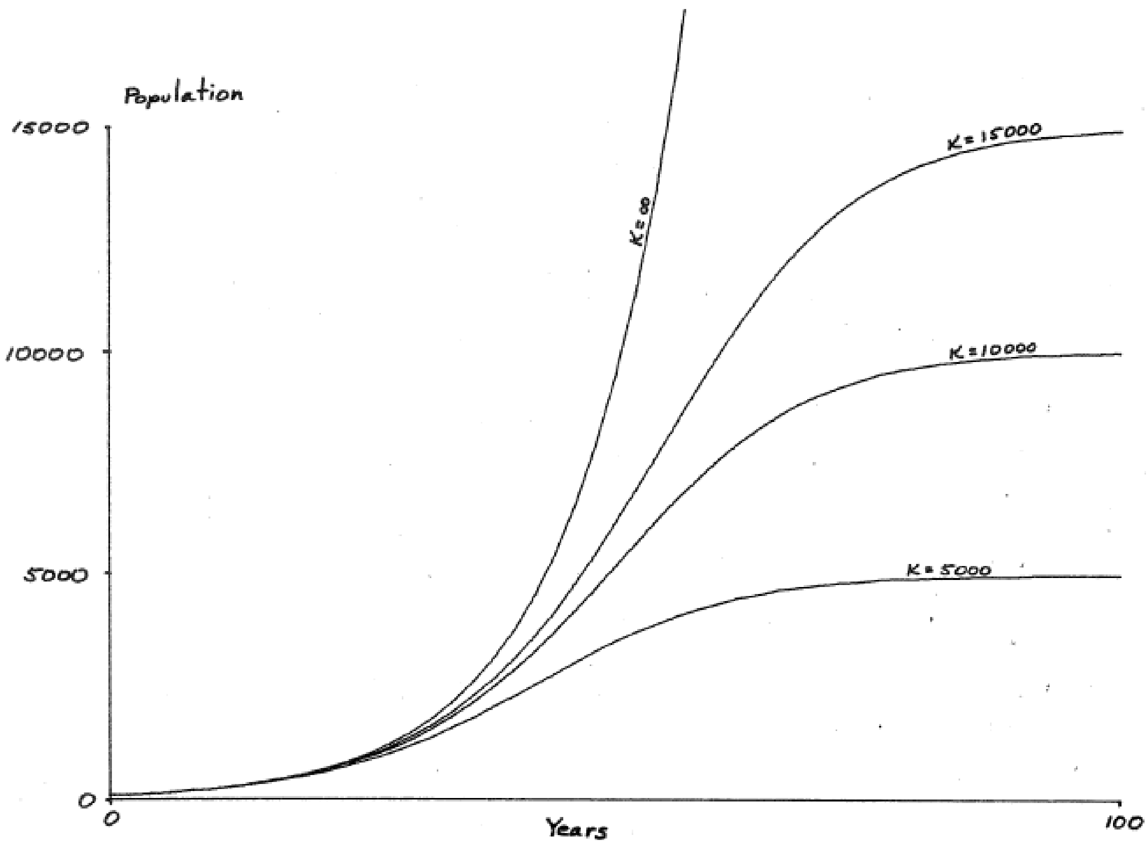

Population
15000
10000
5000
0
0
Years
100
K=∞
K=15000
K=10000
K=5000

Figure 9.3

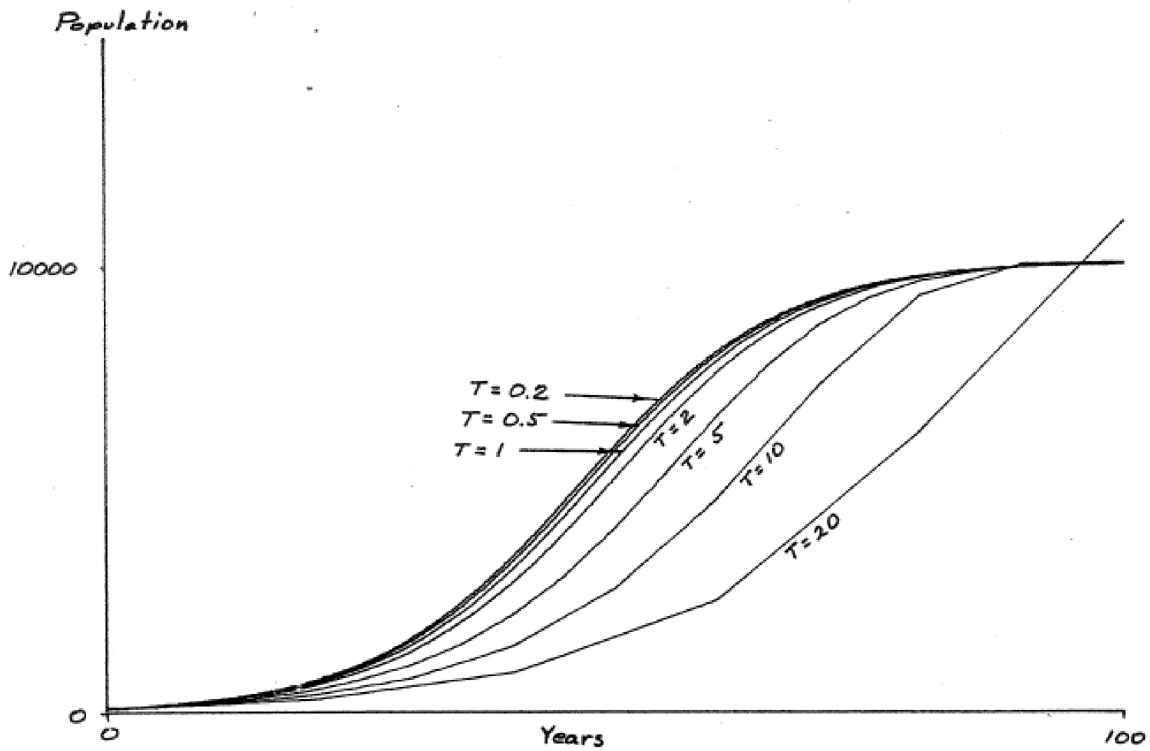


Figure 9.4

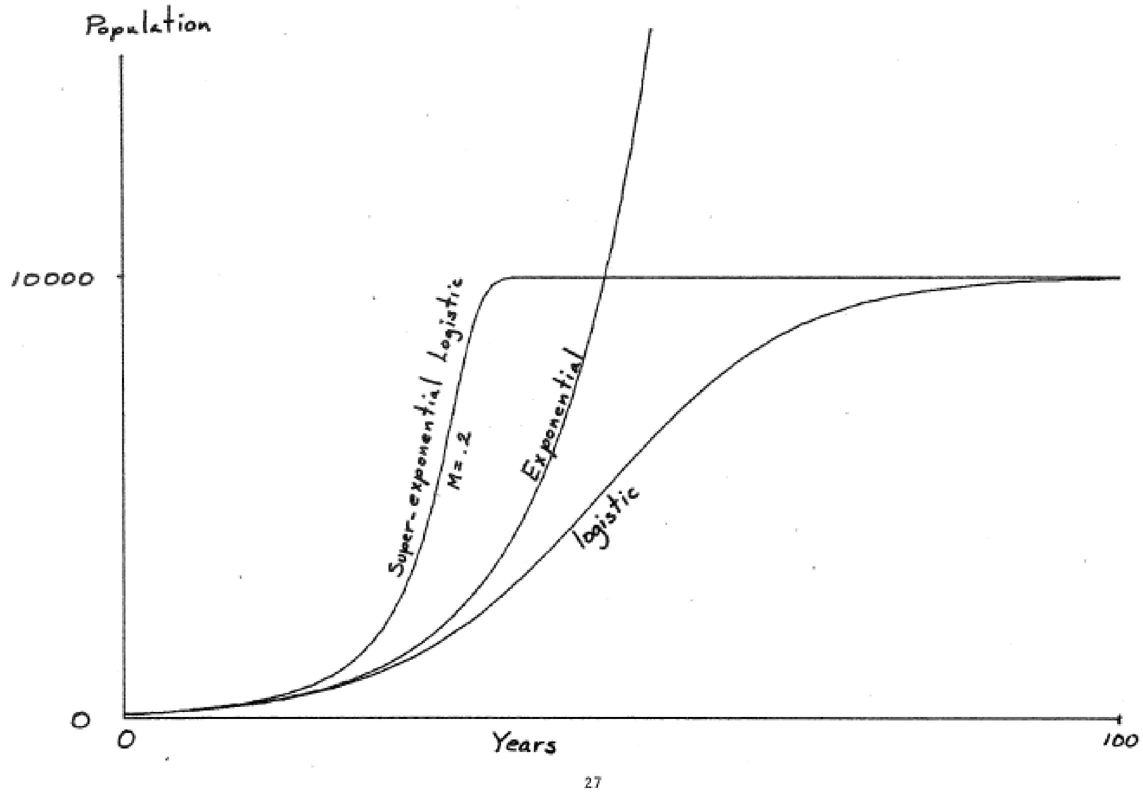

Population
10000
0
0
Years
100
Super-exponential Logistic
M=.2
Exponential
logistic
27

Figure 9.5

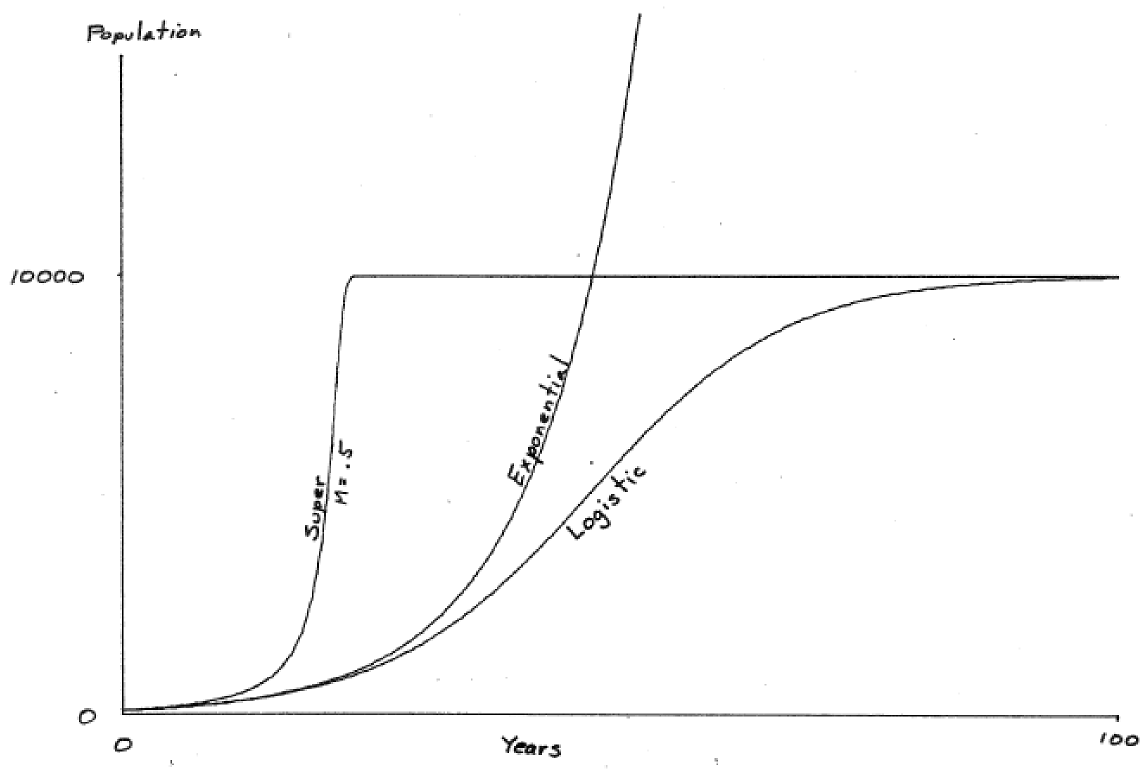

Population
10000
0
0
Years
100
Super
M = .5
Exponential
Logistic

Figure 9.6

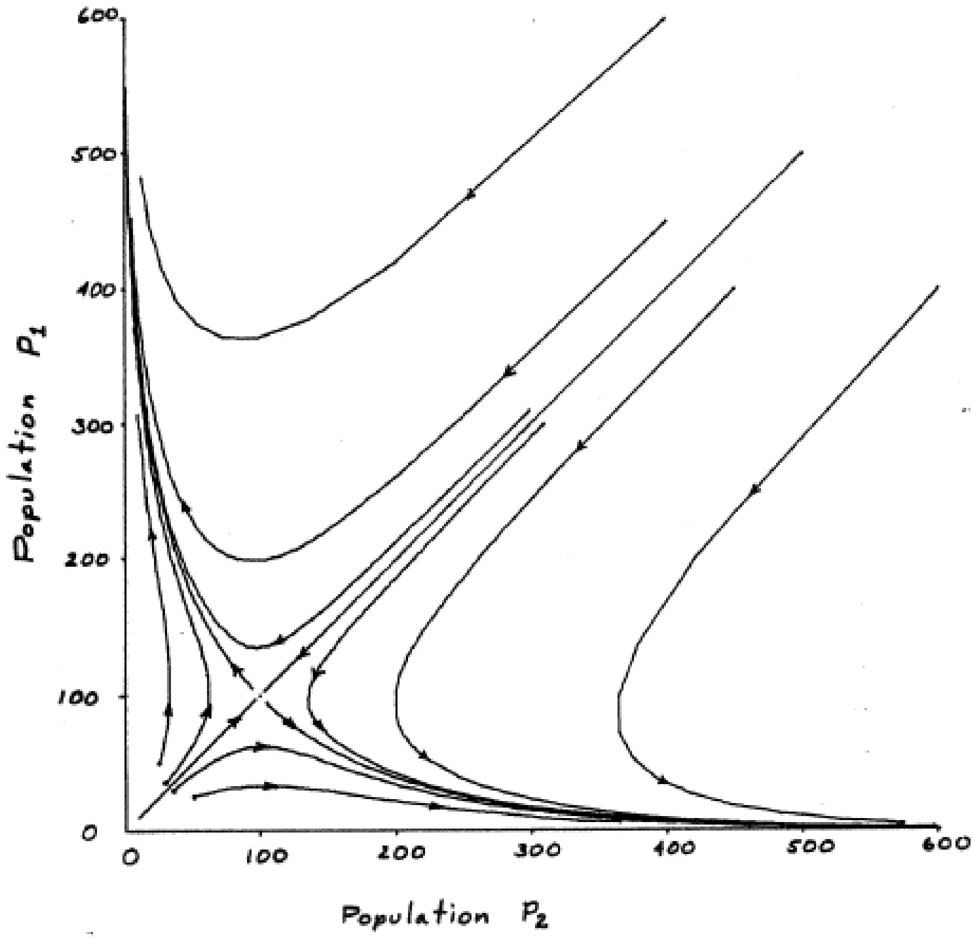

600
500
400
300
200
100
0
Population P_1
0
100
200
300
400
500
600
Population P_2

Figure 9.7

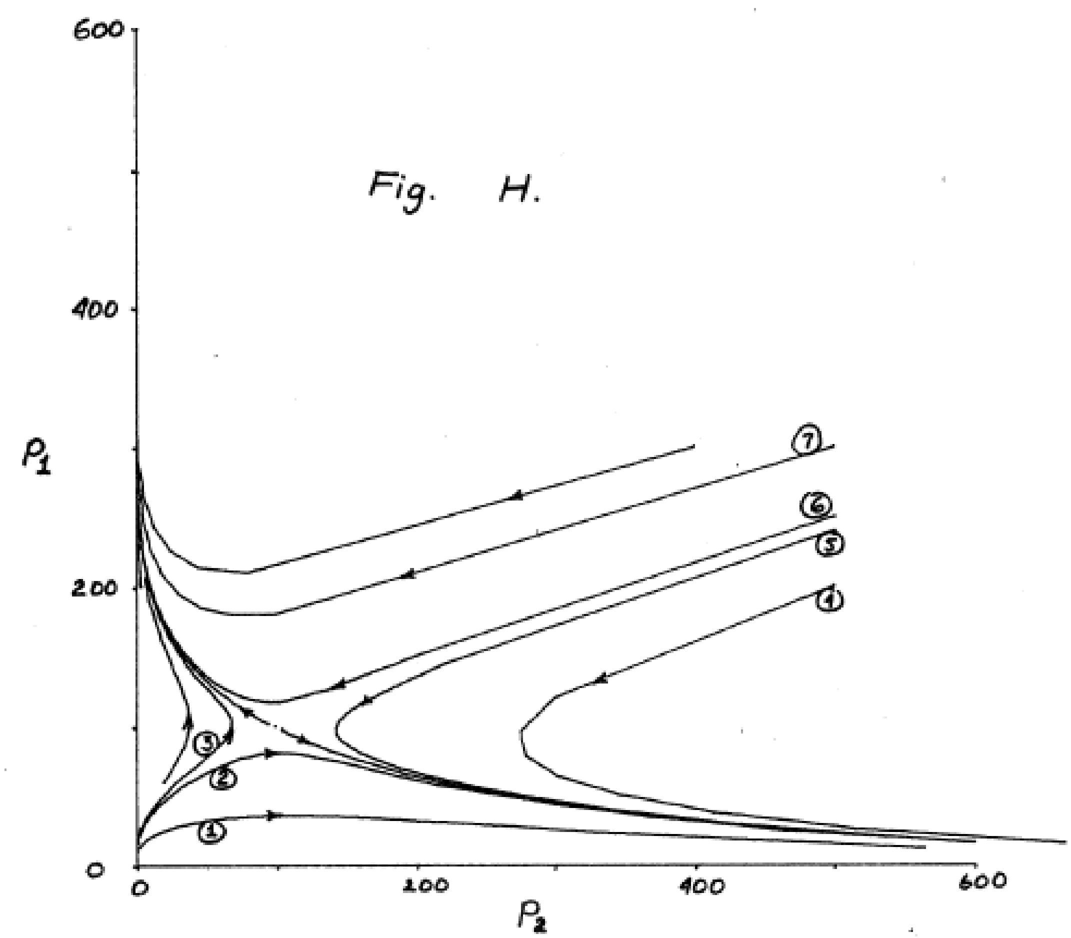


Figure 9.8

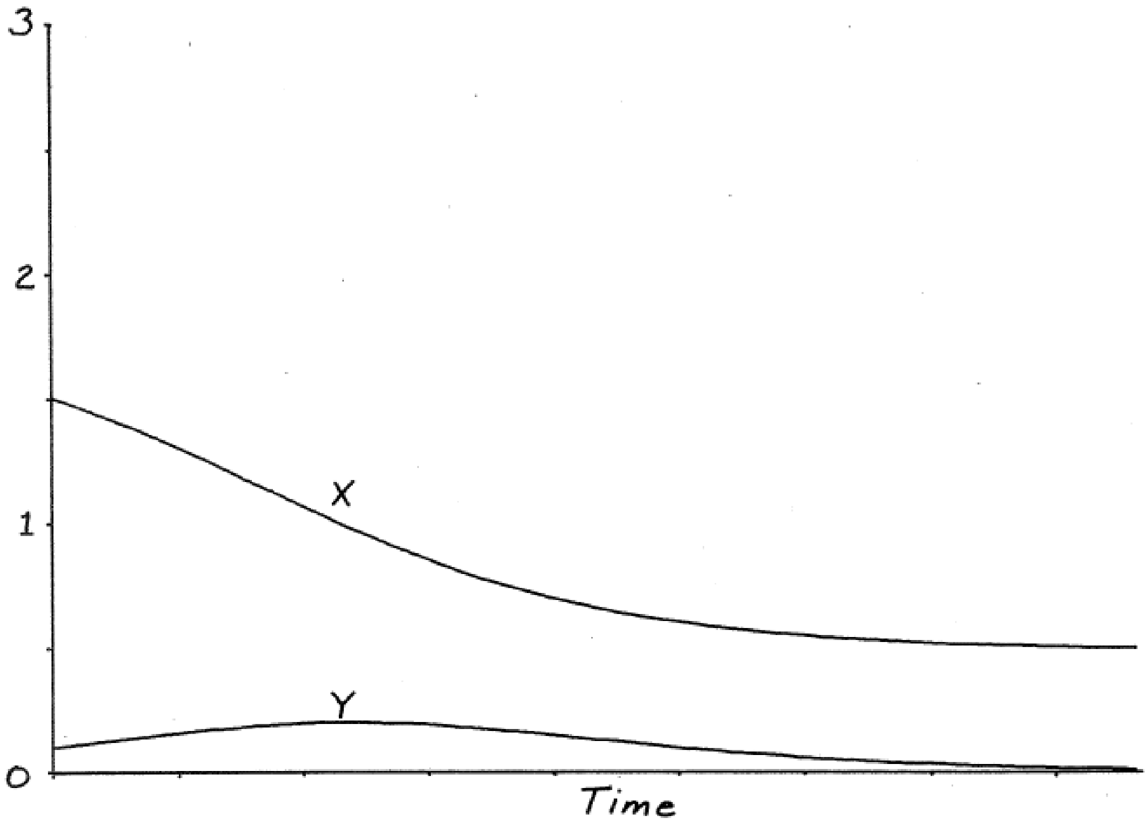


Figure 9.9

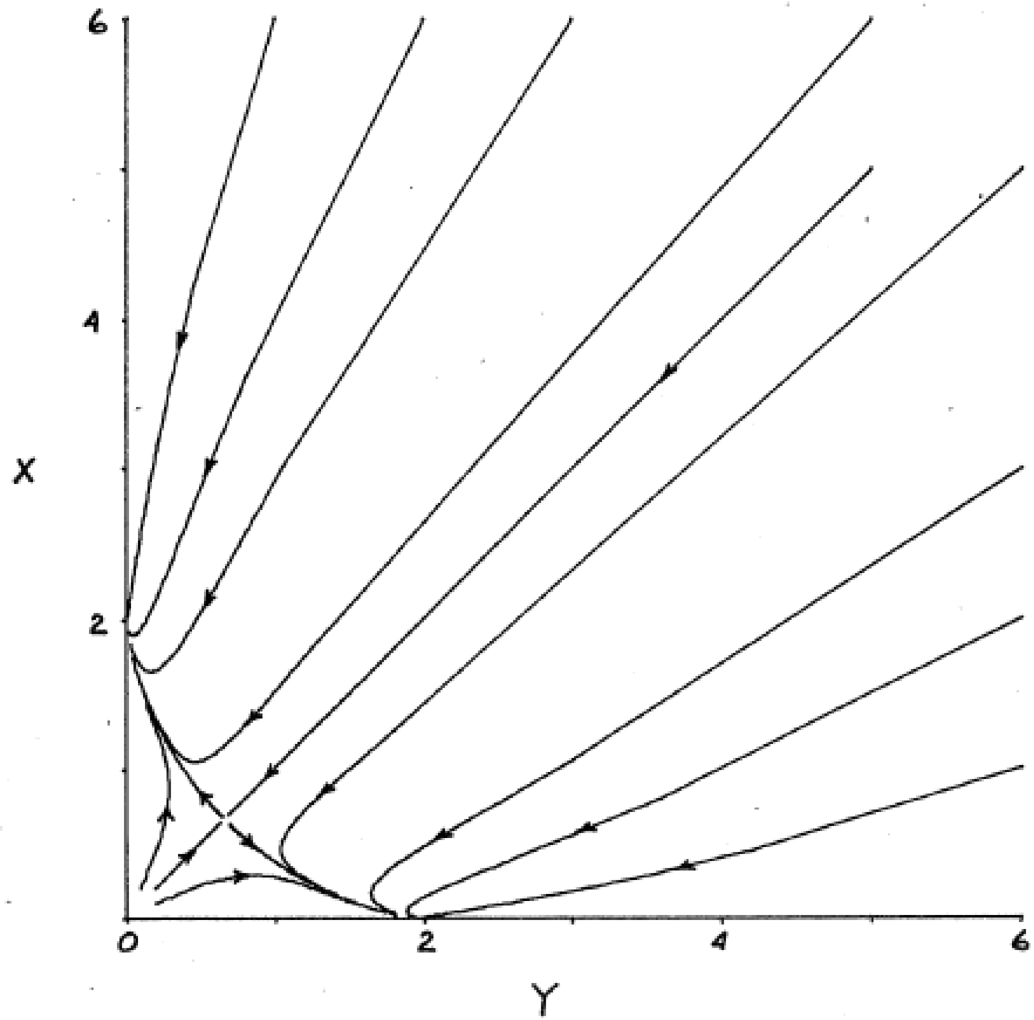


Figure 9.10

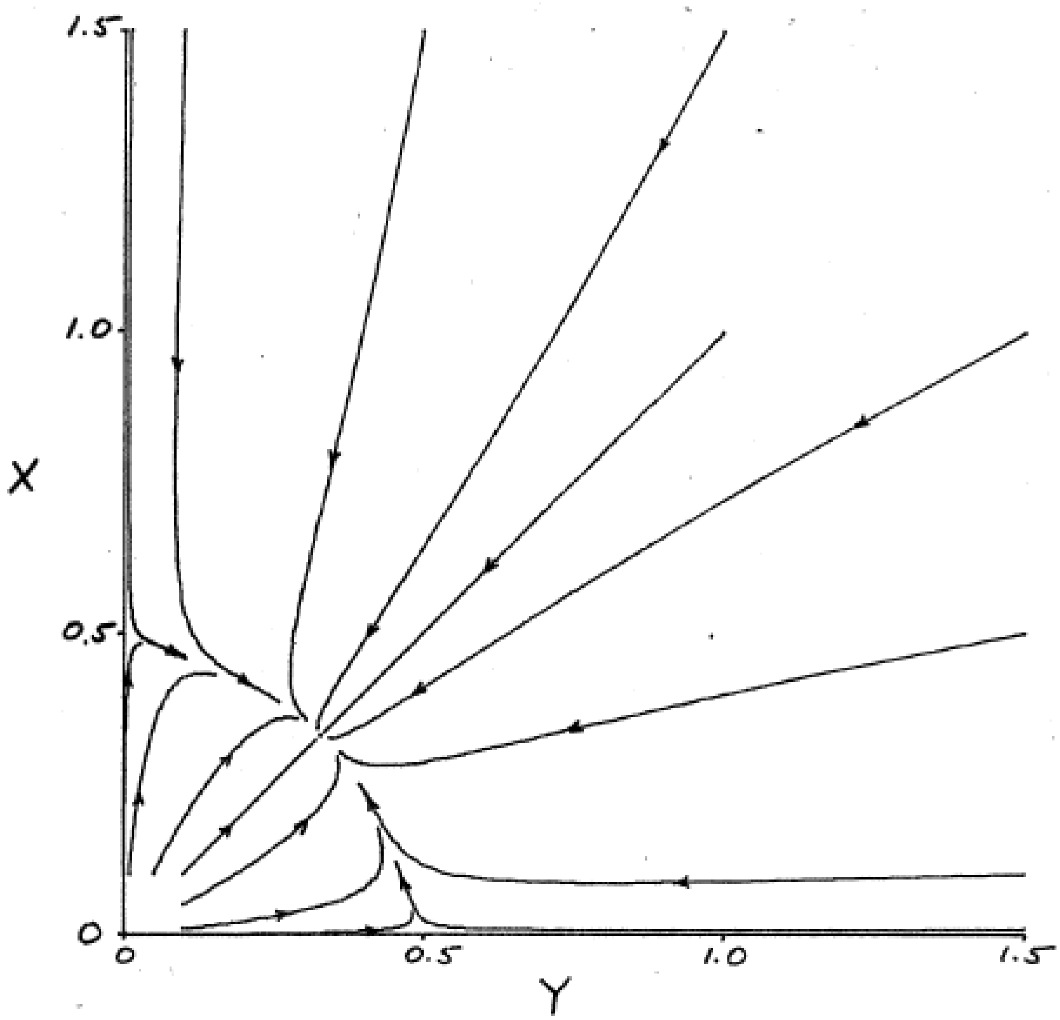


Figure 9.11

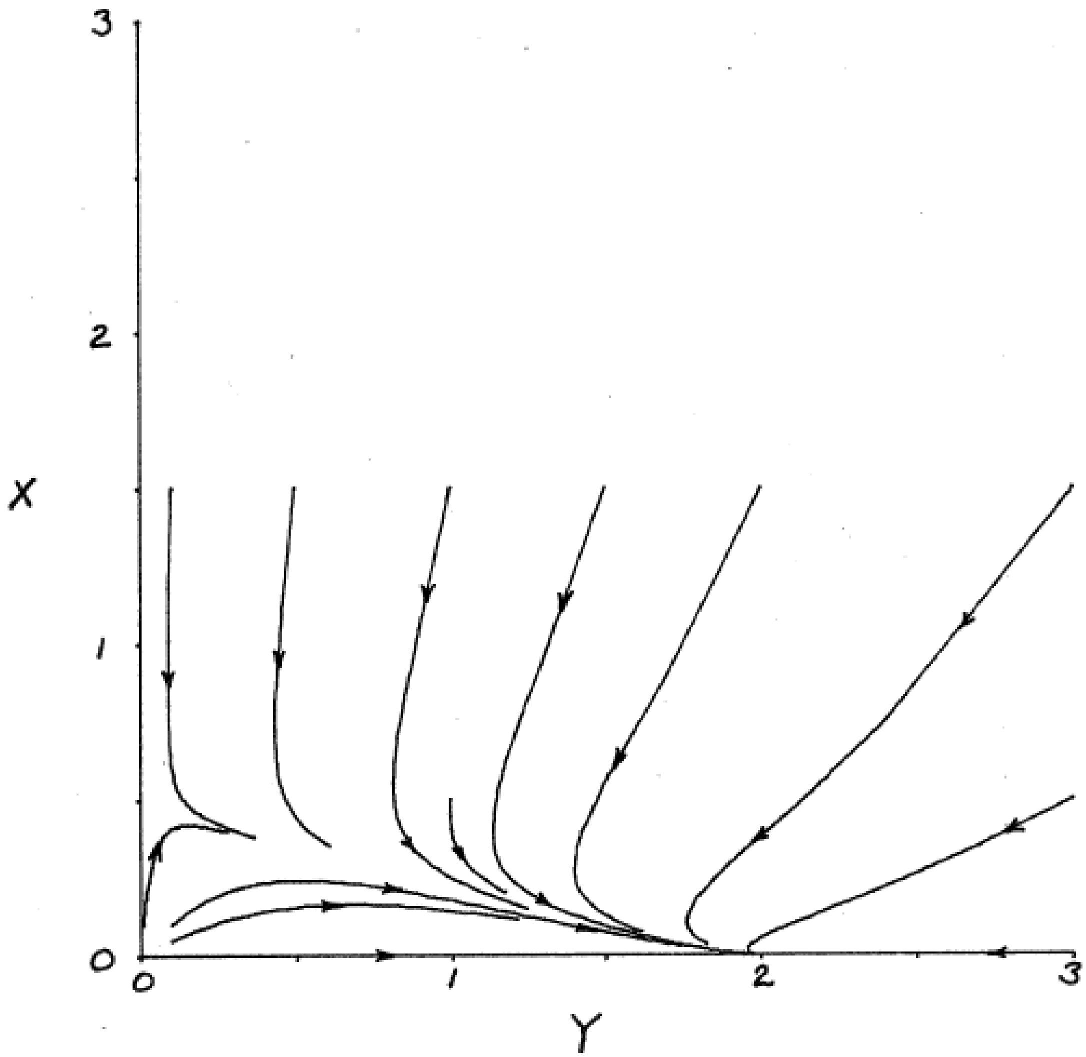


Figure 9.12

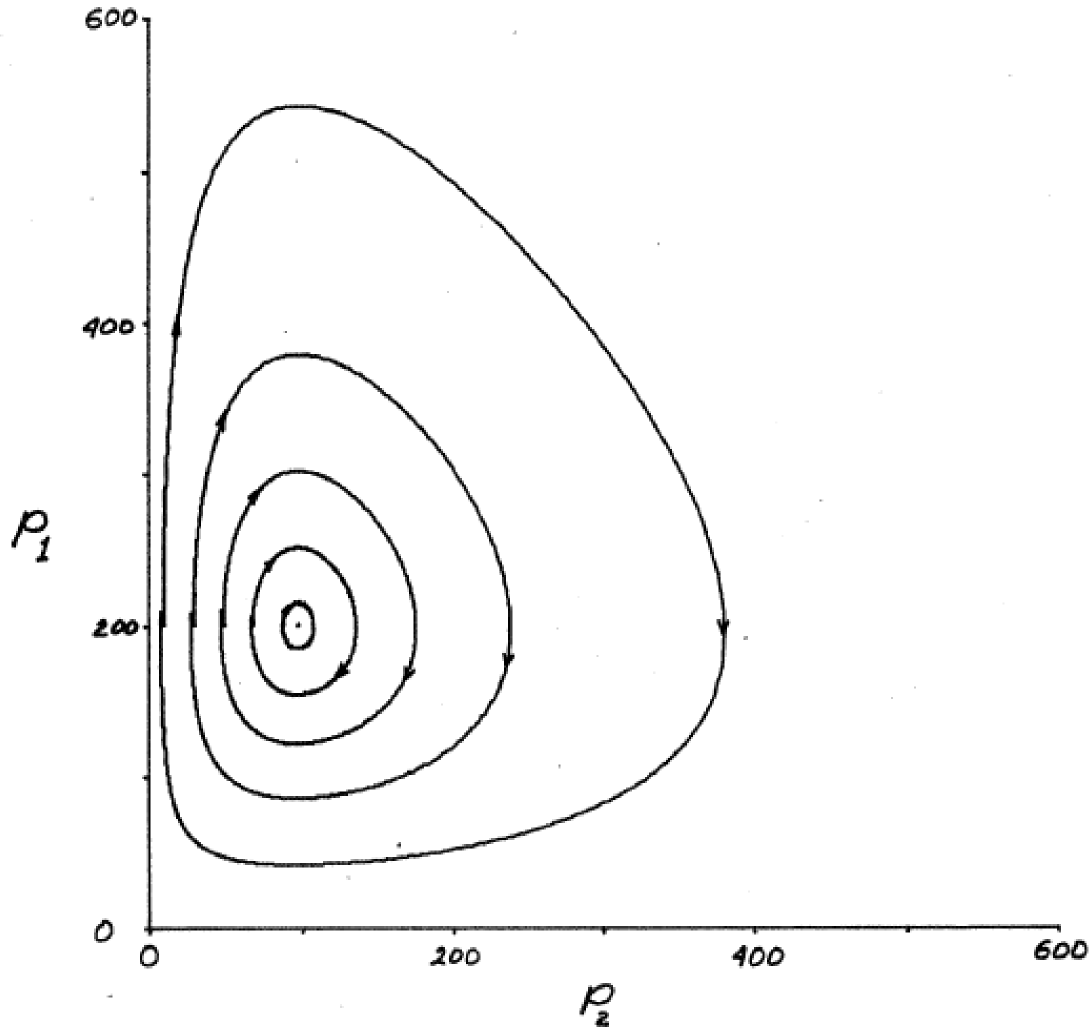
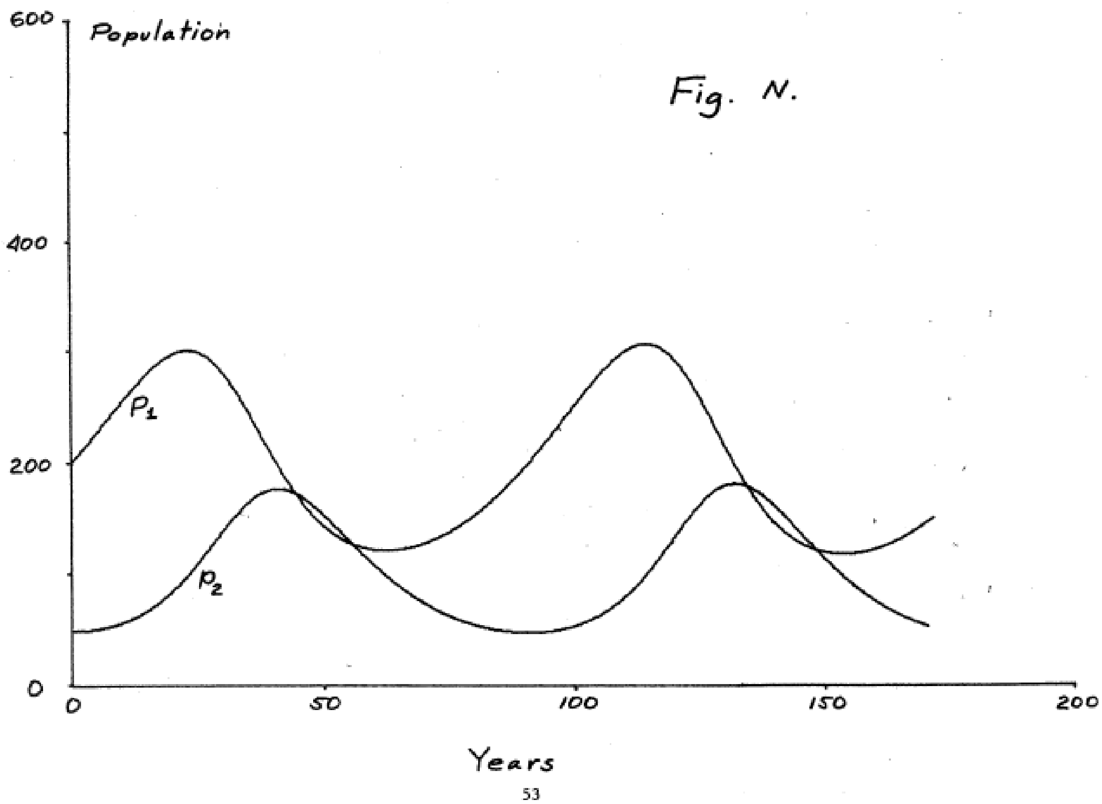


Figure 9.13



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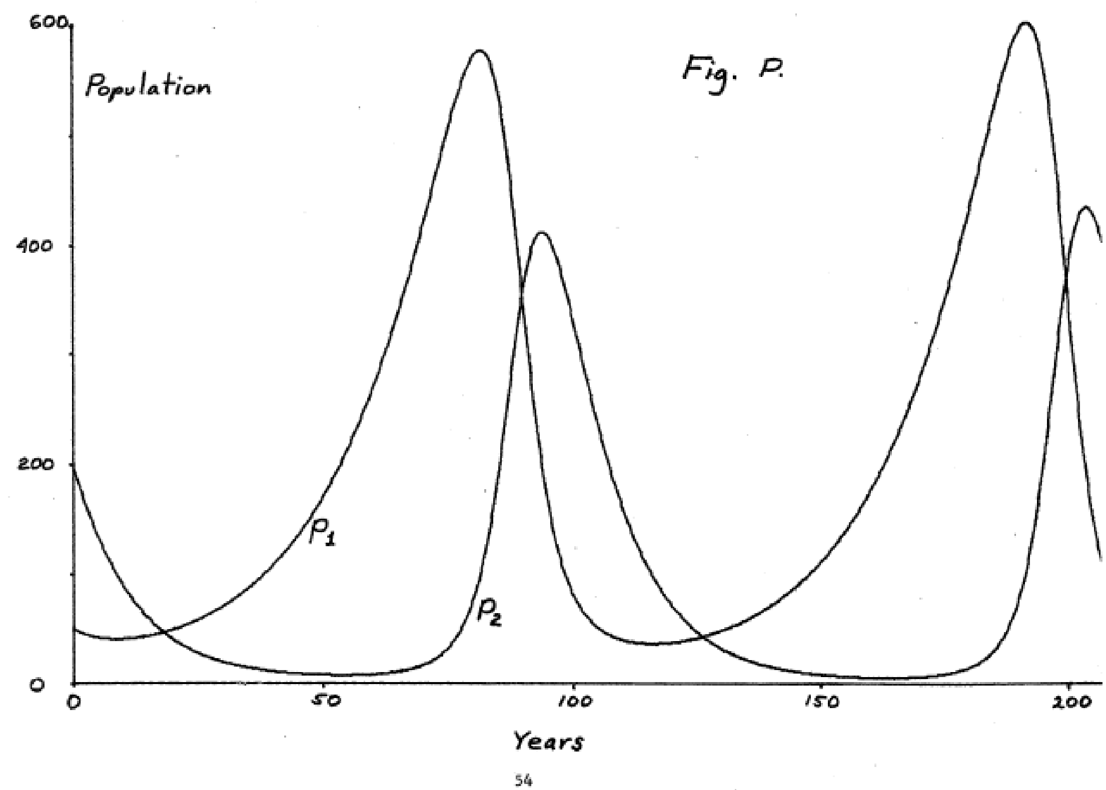

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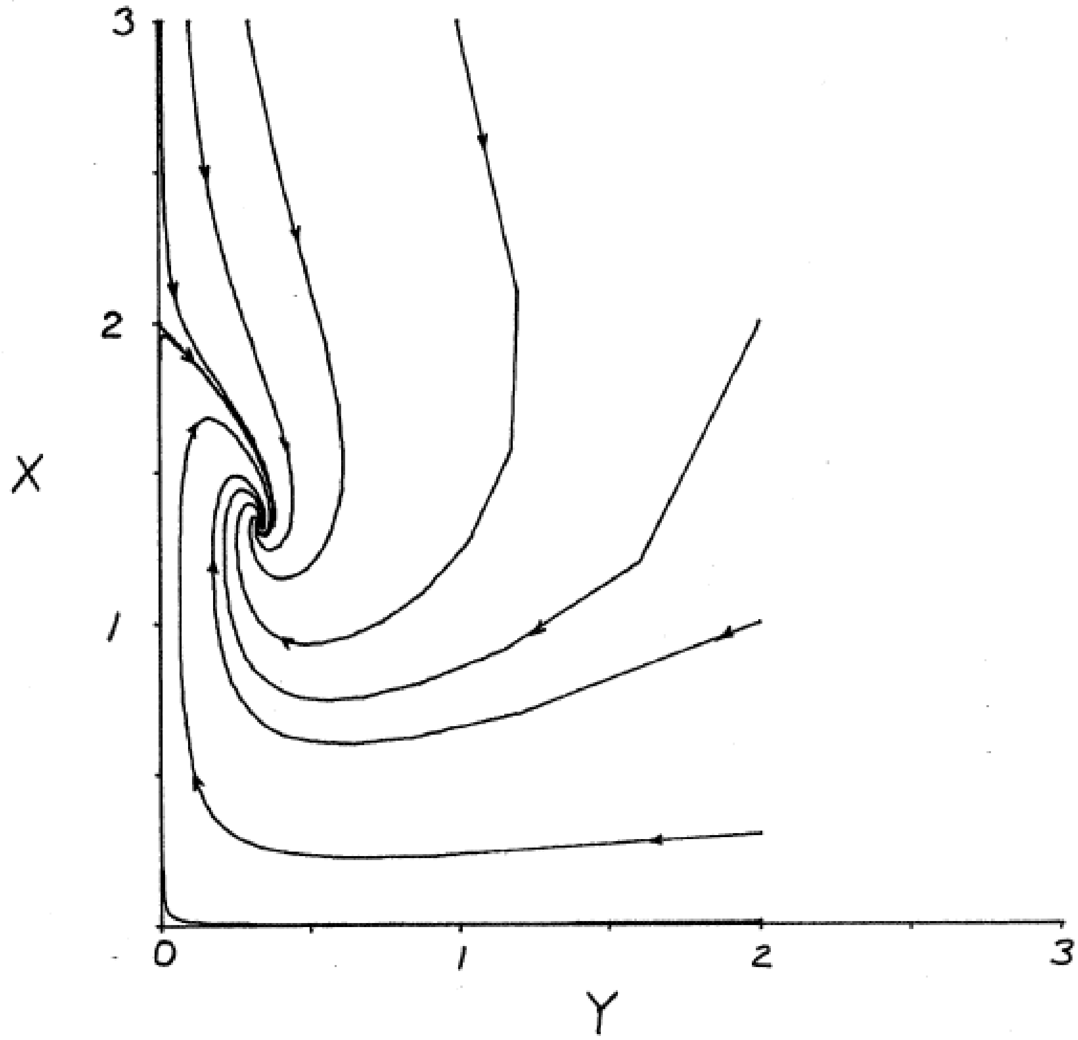

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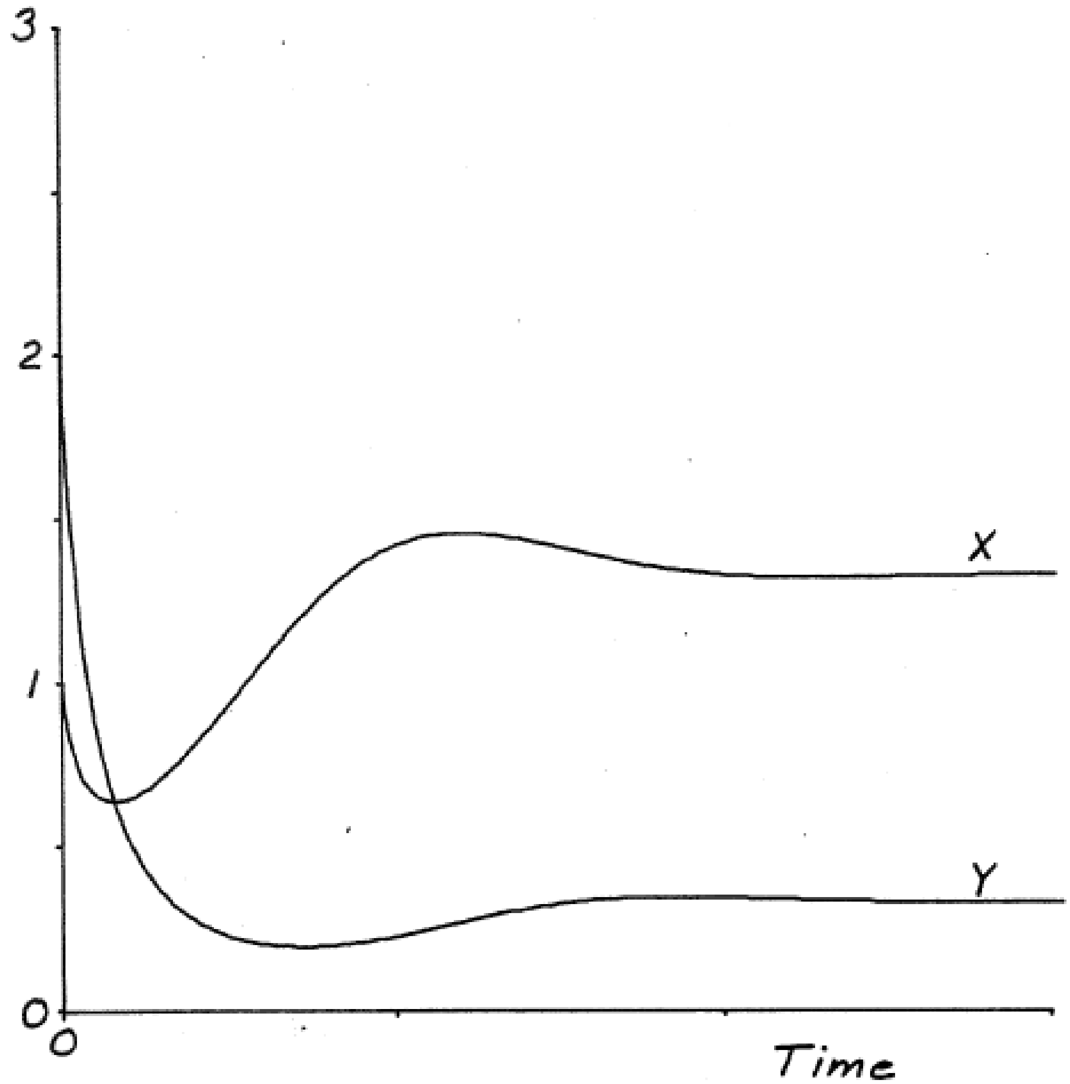

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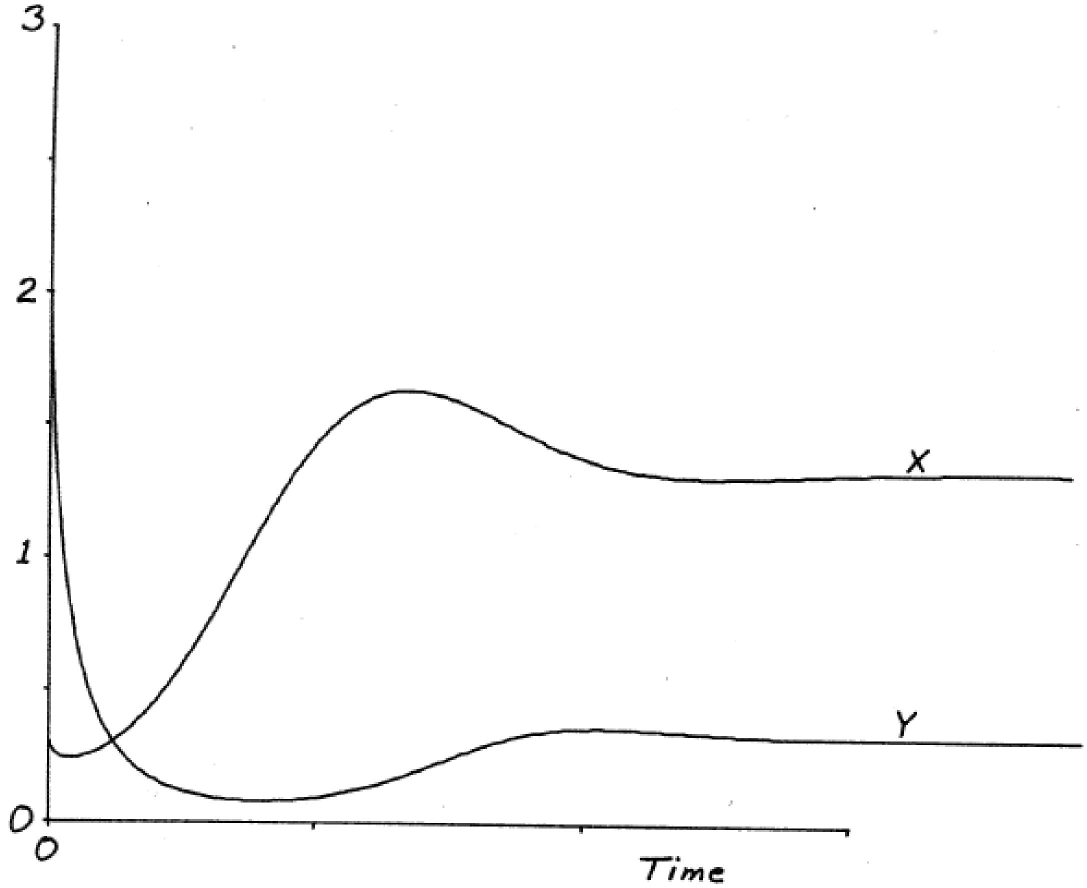

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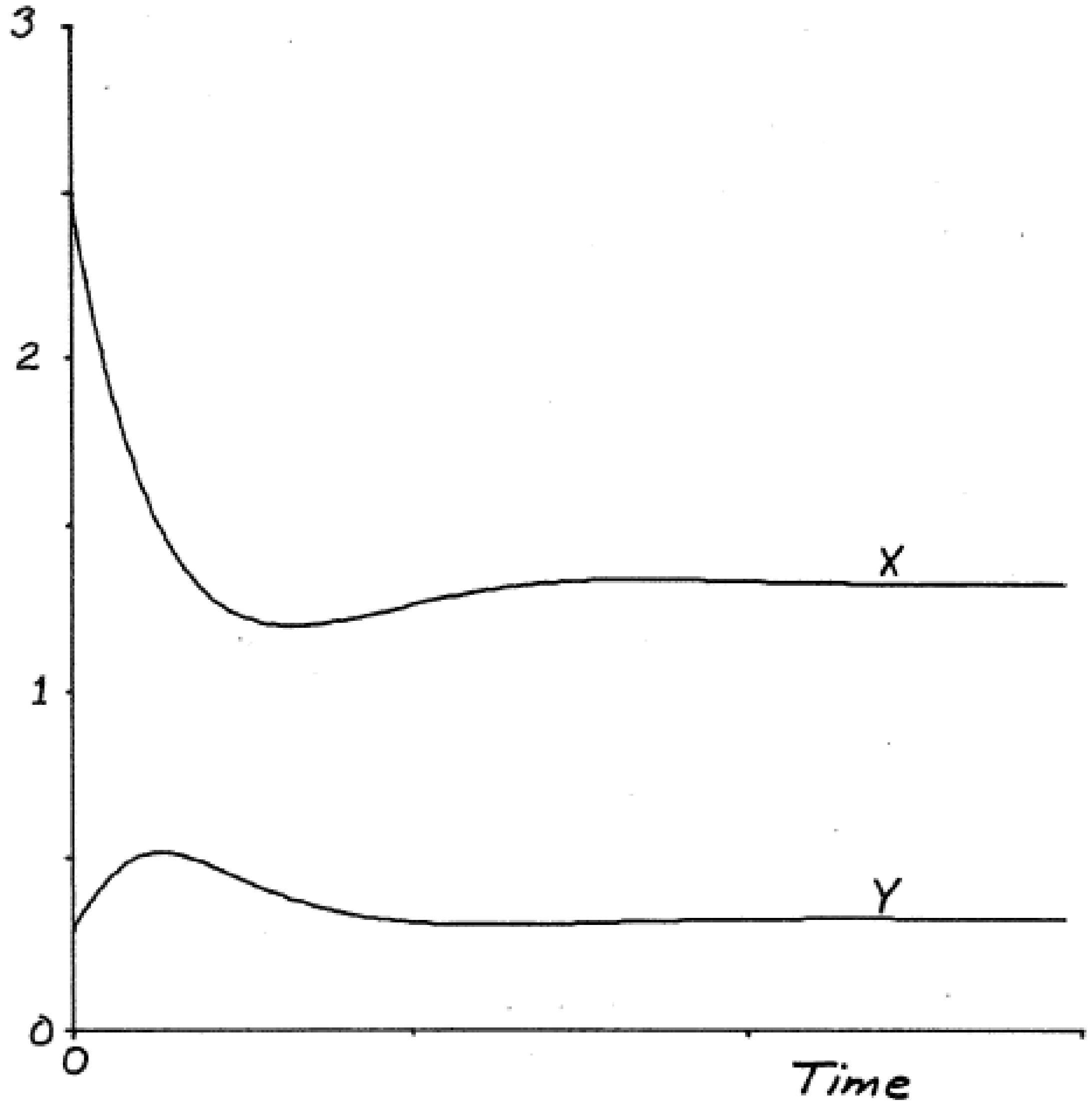

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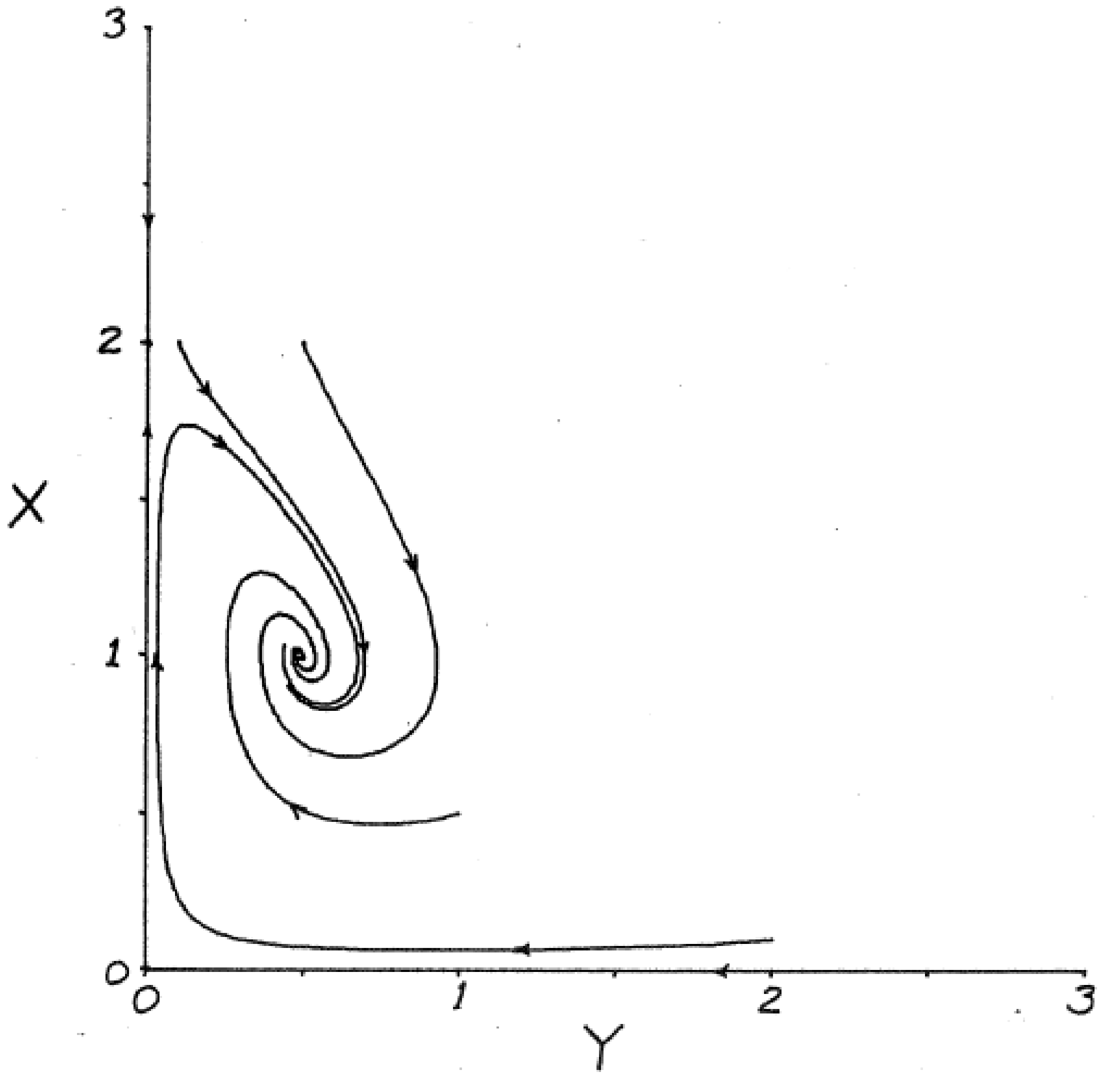

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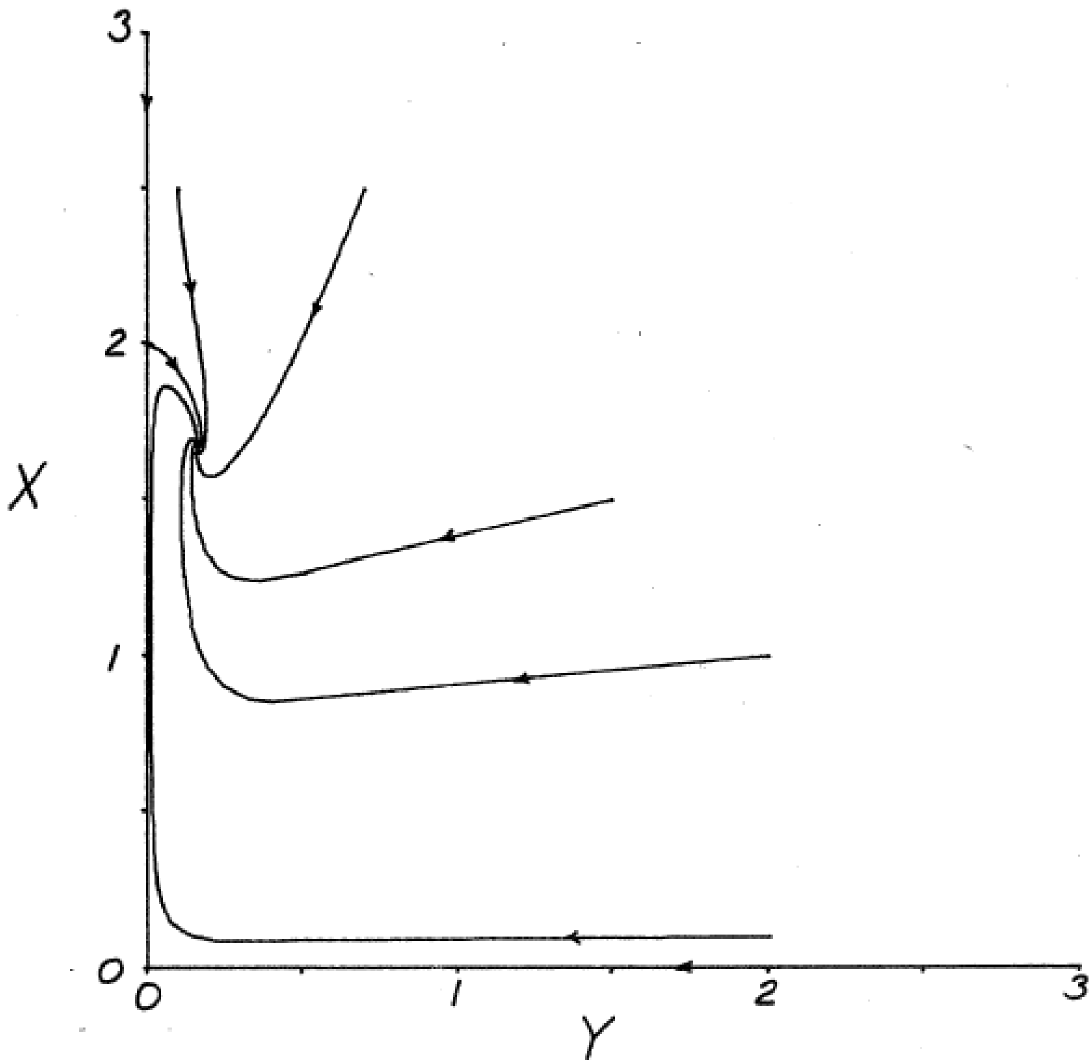

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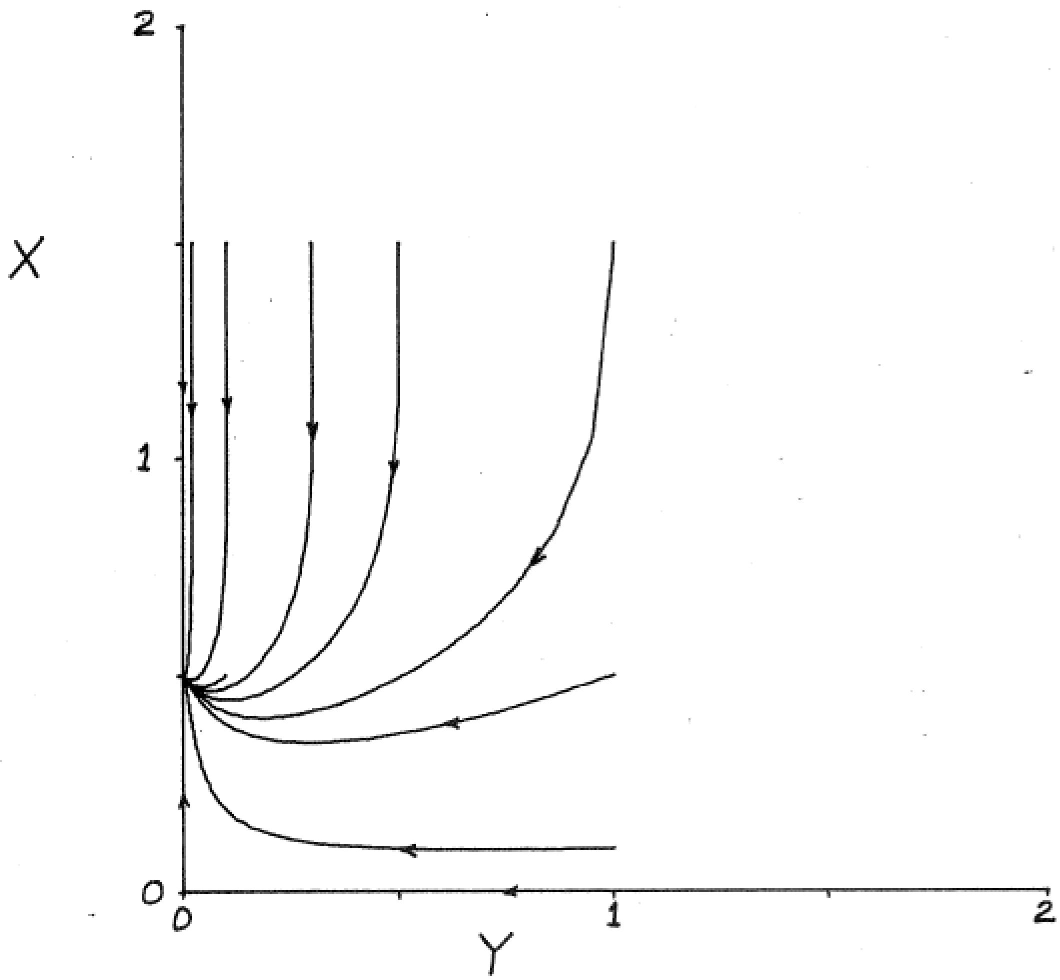

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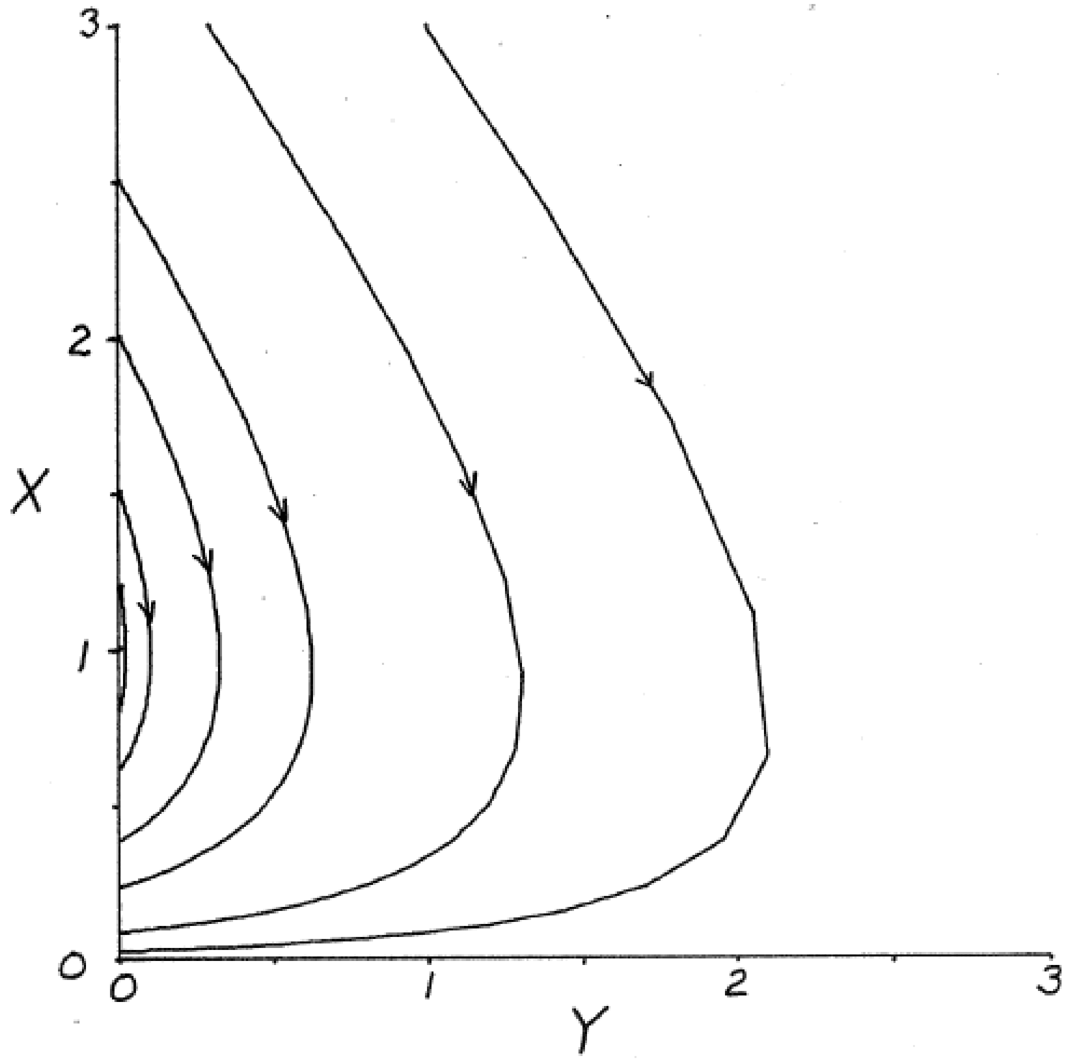

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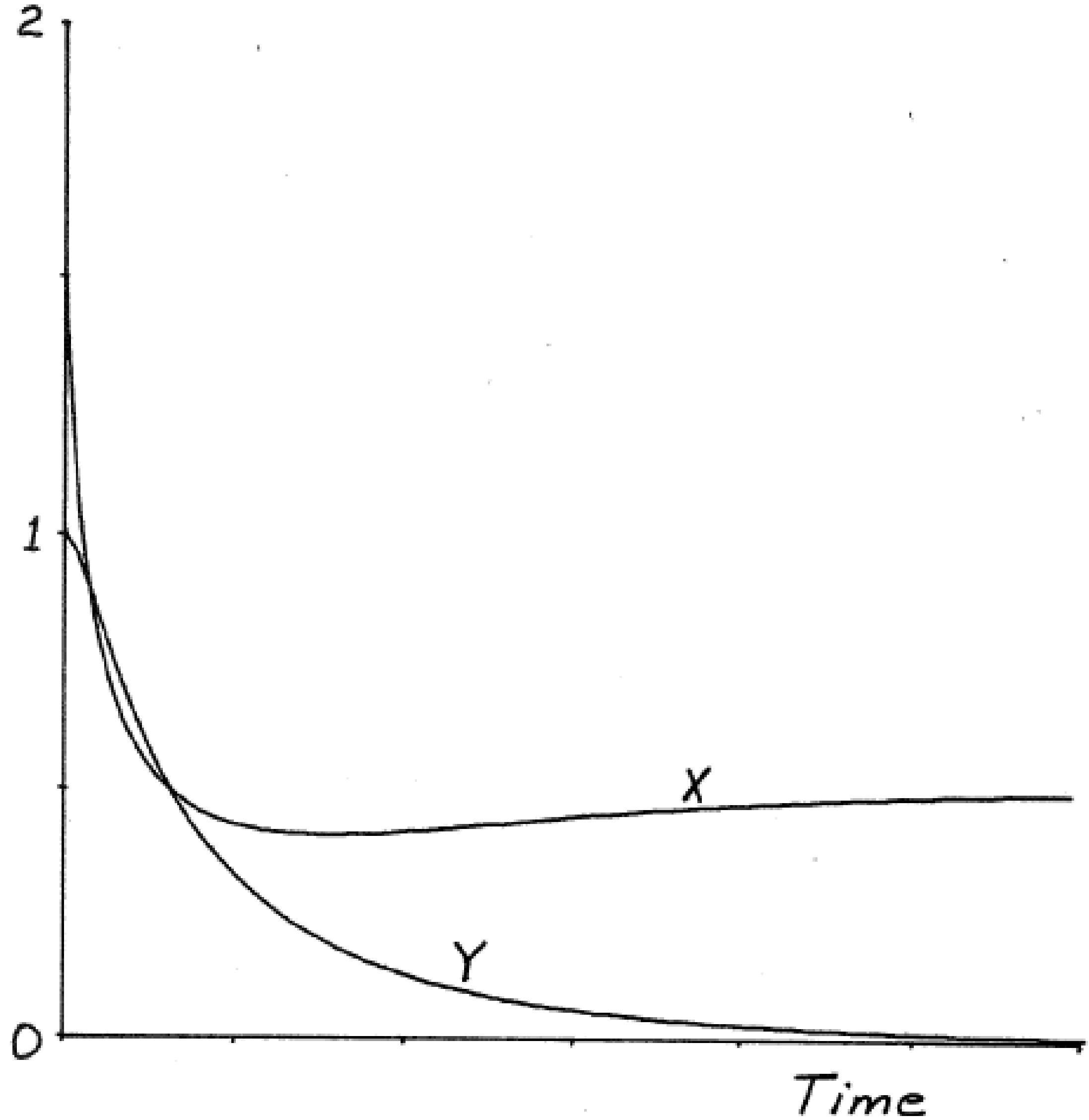

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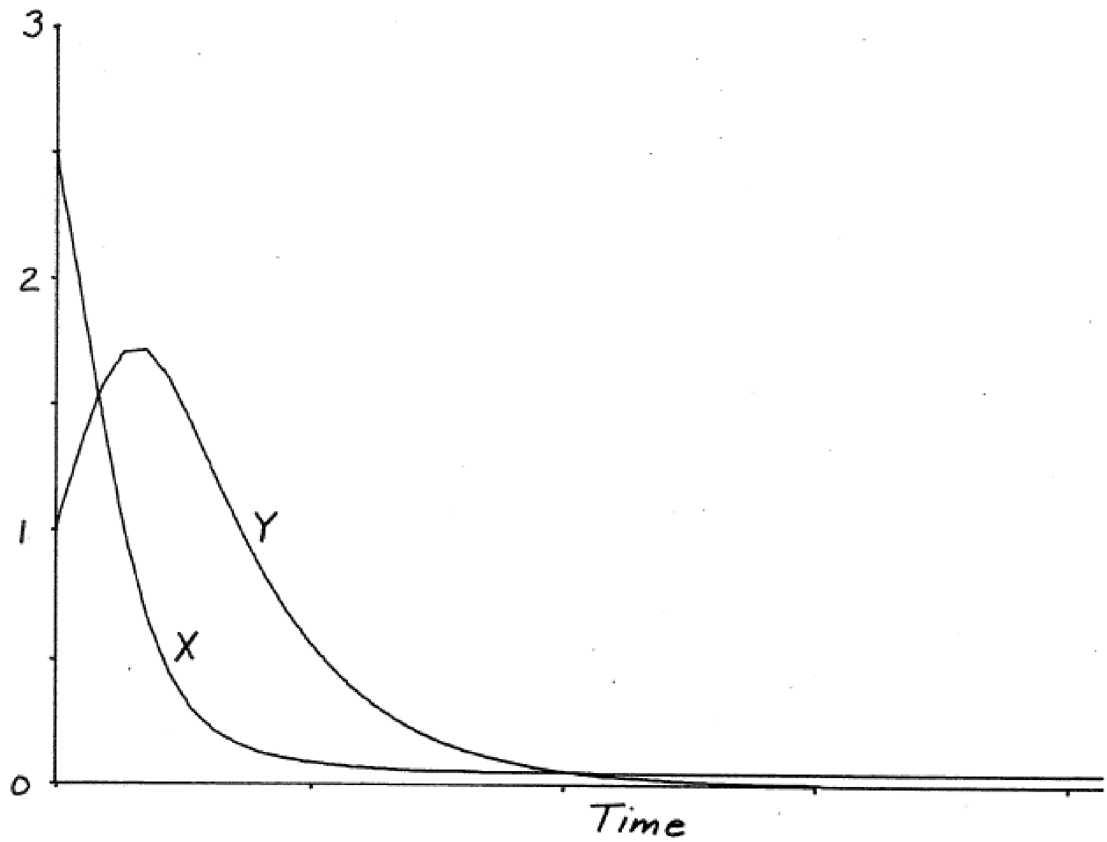

Figure 9.25

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Dynamics of Social Systems
This collection develops the ideas of model, dynamics, and simulation that have been used successfully in science and engineering is a way to be applied in the social sciences. These notes were originally used in a course at Rice University taught to engineers and social scientist in 1973-74.